A GALAXY OF PERIODIC TILINGS

Daniel H. Huson

Night sky: Michael J. Bennett, Wikipedia

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Royal Society of New Zealand
International Leader Catalyst Fellow

Department of Computer Science

Contents

- Topology
- Geometry
- Combinatorics
- Algorithms and Software

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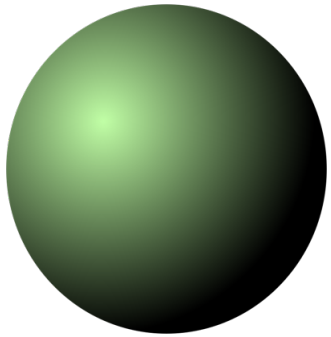
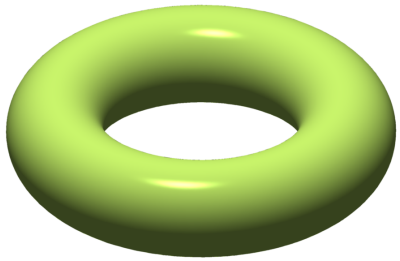
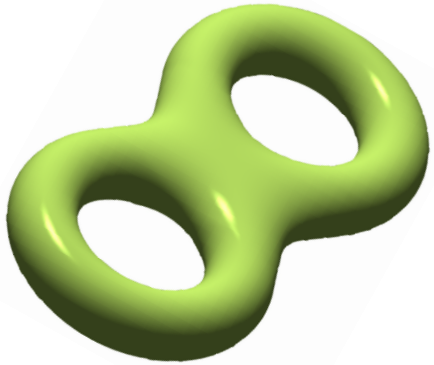
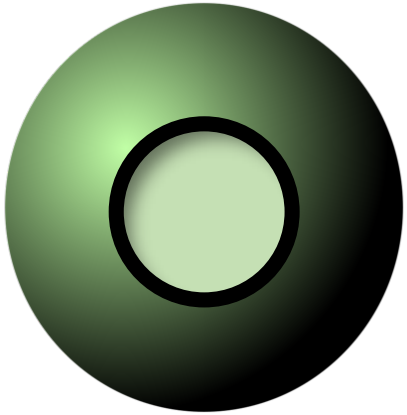
Classification of **orientable** closed surfaces

		Conway:
 Darkdadaah, Wikipedia	sphere	**1**
 Oleg Alexandrov, Wikipedia	torus	**o**
 Oleg Alexandrov, Wikipedia	two handles	**oo**
...	*h* handles	$\underbrace{\mathbf{o\ldots o}}_{h}$

Classification of **non-orientable** closed surfaces



Darkdadaah, Wikipedia

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JoshDif, Wikipedia

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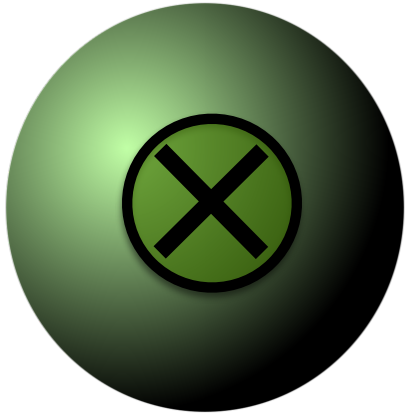


Darkdadaah, Wikipedia

Sphere

Möbius strip

Sphere with cross cap

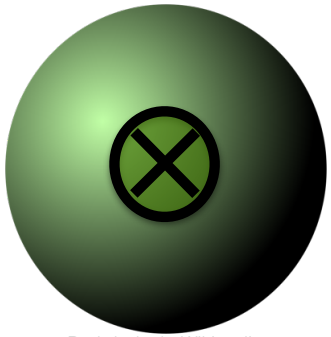
Remove disk

Glue along boundary

Classification of **non-orientable** closed surfaces

sphere & cross cap = projective plane

Conway:



Darkdadaah, Wikipedia

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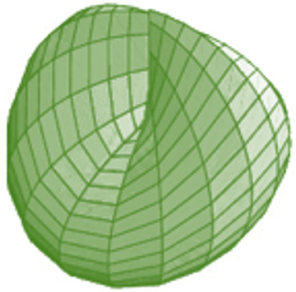
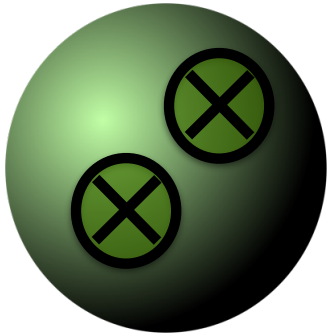


C.H. Sequin & J. Lanier, 2007

x

sphere & two cross caps = Klein bottle



Darkdadaah, Wikipedia

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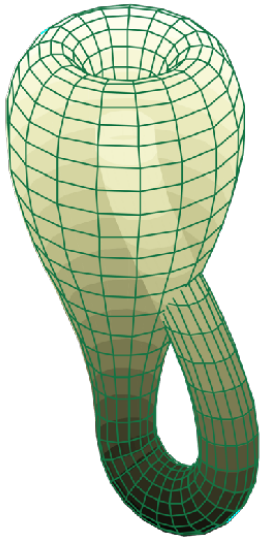
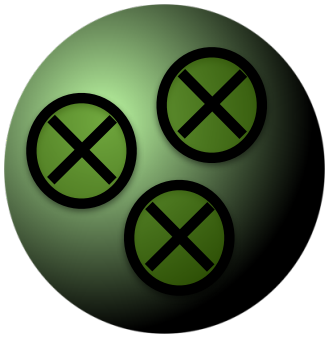


Tttrung, Wikipedia

Connected sum of two projective planes

xx

Classification of **non-orientable** closed surfaces



Darkdadaah, Wikipedia

Conway:

xxx

...

$\underbrace{\mathbf{x\ldots x}}_{k}$

Classification of closed surfaces

Theorem

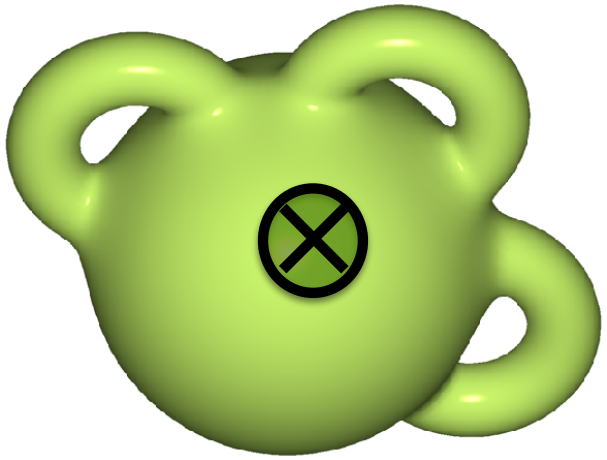
Any connected closed surface is either a

- sphere, **1**
- sphere with $h > 0$ handles, or $\underbrace{\mathbf{o}\ldots\mathbf{o}}_{h}$
- sphere with $k > 0$ cross caps. $\underbrace{\mathbf{x}\ldots\mathbf{x}}_{k}$

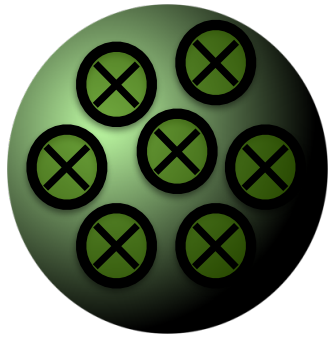
Classification of closed surfaces

Do not need to combine **both** handles and cross caps

- Non-orientable surface:
- Can replace *one* handle by *two* cross caps:



Oleg Alexandrov, Wikipedia

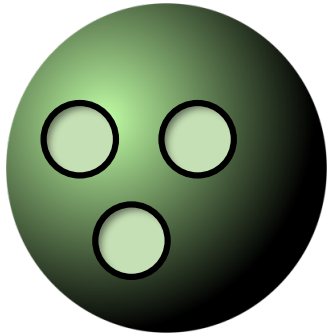


Darkdadaah, Wikipedia

oooX ⟷ **XXXXXXX**

Surfaces with boundary

Example: sphere with three disks removed



Darkdadaah, Wikipedia

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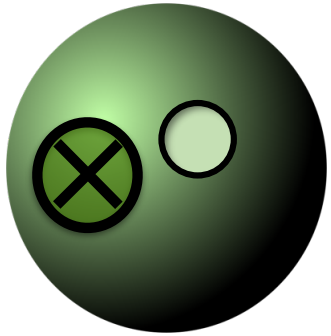
Example: two-handled sphere
with four disks removed

Oleg Alexandrov, Wikipedia

oo****

Surfaces with boundary



Darkdadaah, Wikipedia

$*\mathbf{x}$

Möbius strip

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Classification of surfaces

Theorem

Any connected surface, closed or with boundary, has Conway (orbifold) symbol

- **1** — sphere
- $\underbrace{\mathbf{**\ldots}}_{b \geq 1}$, — sphere with *b* disks removed
- $\underbrace{\mathbf{oo\ldots}}_{h \geq 1}\underbrace{\mathbf{**\ldots}}_{b \geq 0}$, or — sphere with *h* handles, and *b* disks removed
- $\underbrace{\mathbf{**\ldots}}_{b \geq 0}\underbrace{\mathbf{xx\ldots}}_{k \geq 1}$. — sphere with *k* cross caps, and *b* disks removed

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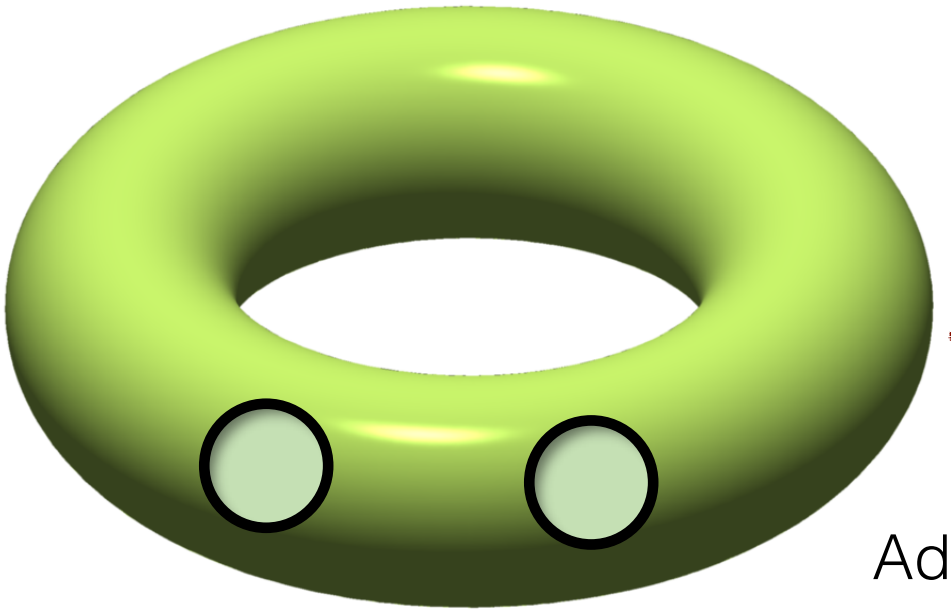
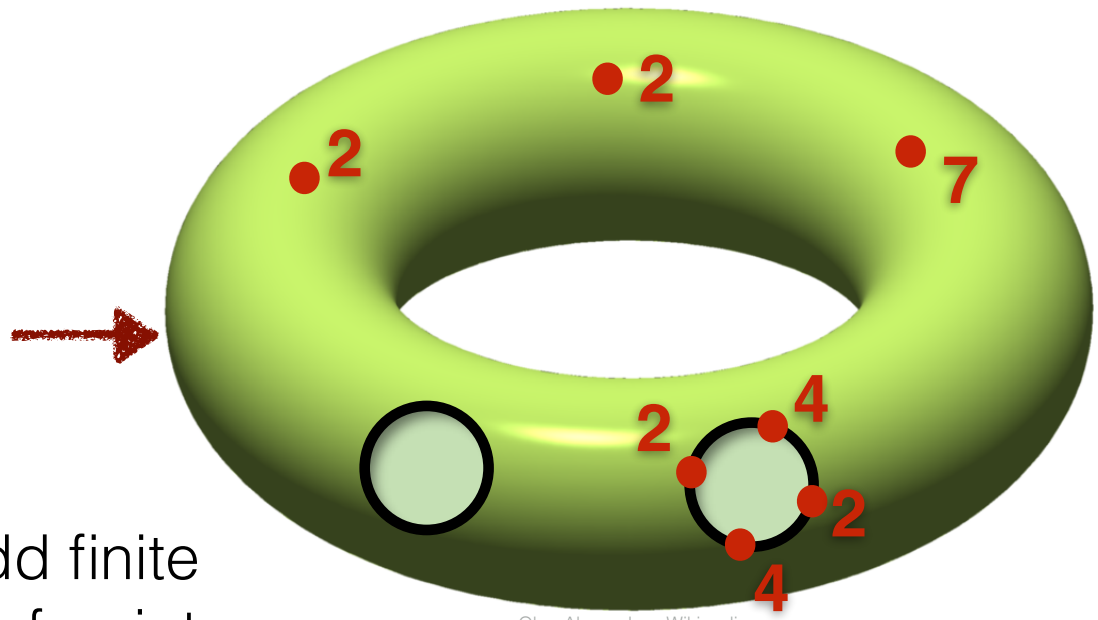
Two-dimensional orbifolds

Orbifold = orbit manifold, W. Thurston



Oleg Alexandrov, Wikipedia



Oleg Alexandrov, Wikipedia

Add finite
set of points,
with labels ≥ 2

o**

o2272424**

J.H. Conway and D.H.H. (2002) Orbifold notation for two-dimensional groups.

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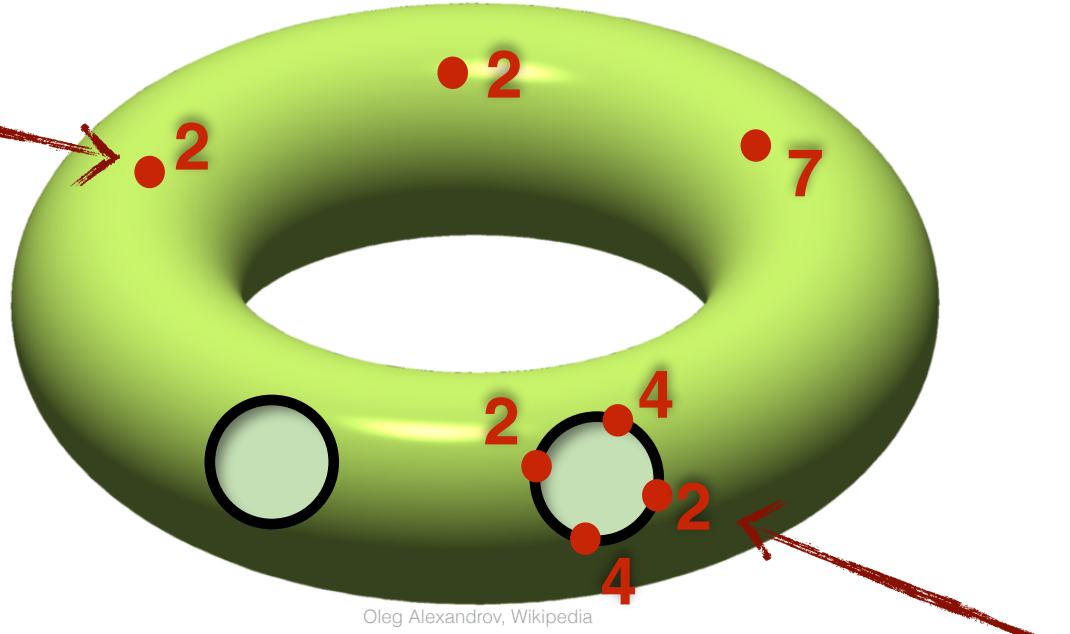
Two-dimensional orbifolds

Gyration



Oleg Alexandrov, Wikipedia

Corner

o2272424**

An orbifold is a topological space together with an "orbifold structure", but we skip the details here.

J.H. Conway and D.H.H. (2002) Orbifold notation for two-dimensional groups.

Conway's orbifold notation

$$\underbrace{o\ldots o}_{\text{handles}}\underbrace{ABC\ldots}_{\text{gyrations}}{}^{*}\underbrace{abc\ldots}_{\text{corners}}{}^{*}\underbrace{rpq\ldots}_{\text{corners}}{}^{*}\ldots\underbrace{x\ldots x}_{\text{cross caps}}$$

J.H. Conway and D.H.H. (2002) Orbifold notation for two-dimensional groups.

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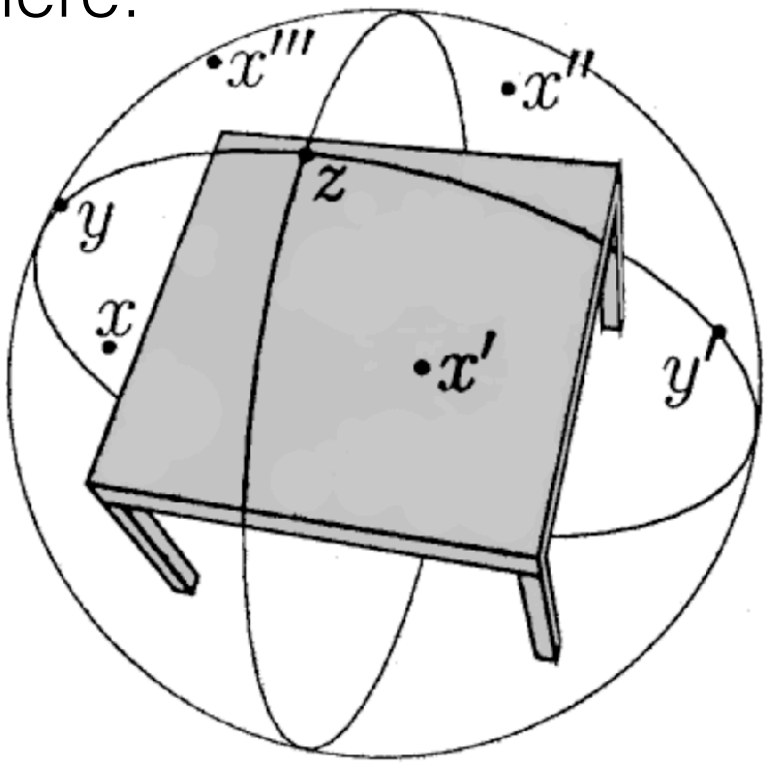
Symmetry groups

- We will consider 2D symmetry groups with compact fundamental domain.

- Example: symmetries of an object, acting on an enclosing sphere:



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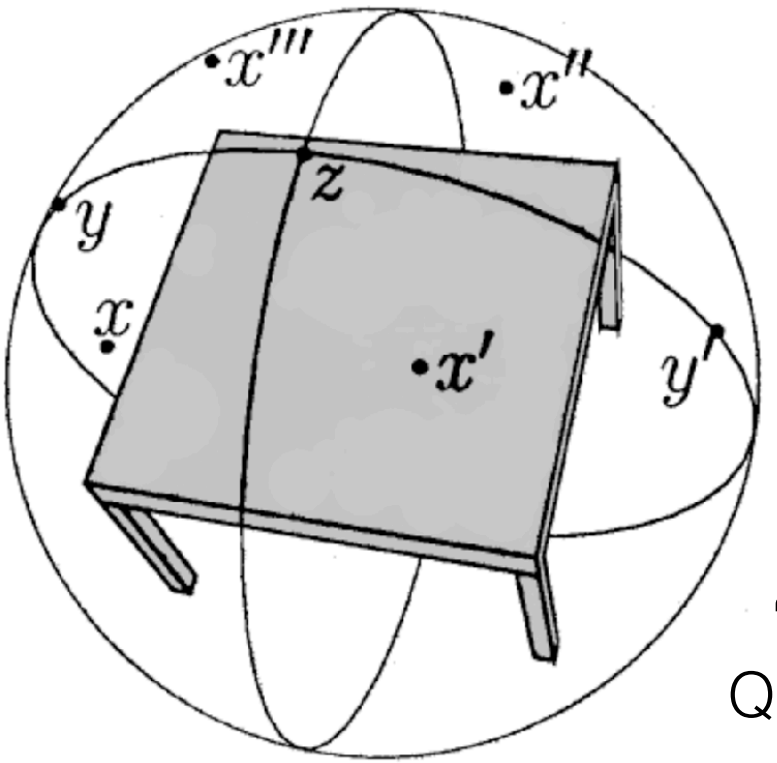
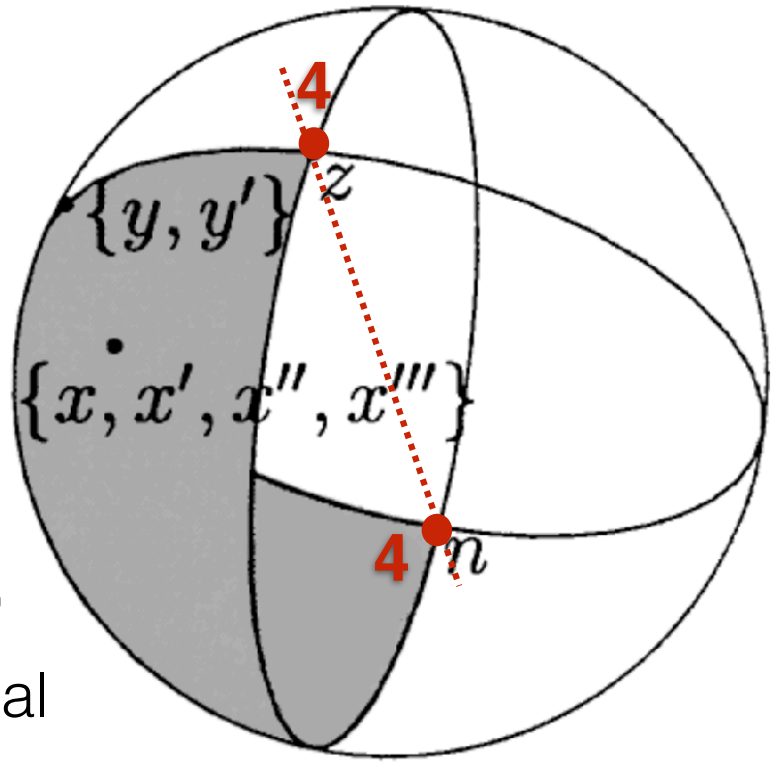
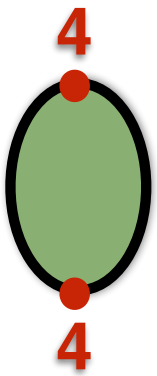
Symmetry groups and orbifolds





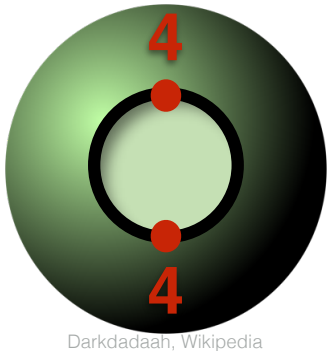
"surface / group"
Quotient topological
space of orbits

sphere **1**



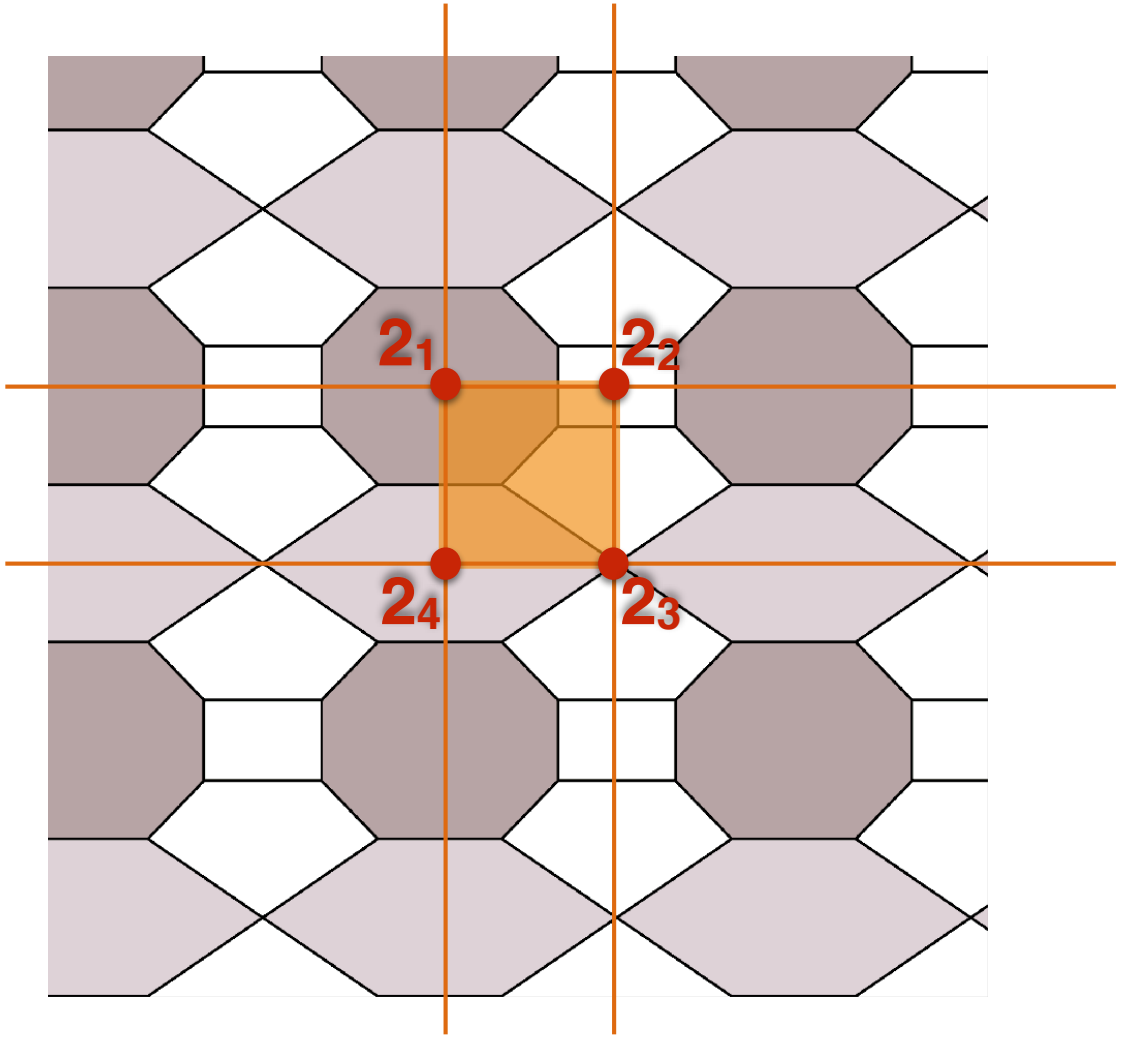
***44**

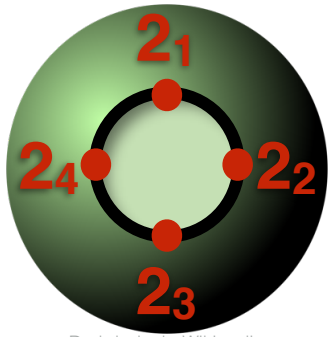
or:



Darkdadaah, Wikipedia

Symmetry groups and orbifolds





Darkdadaah, Wikipedia

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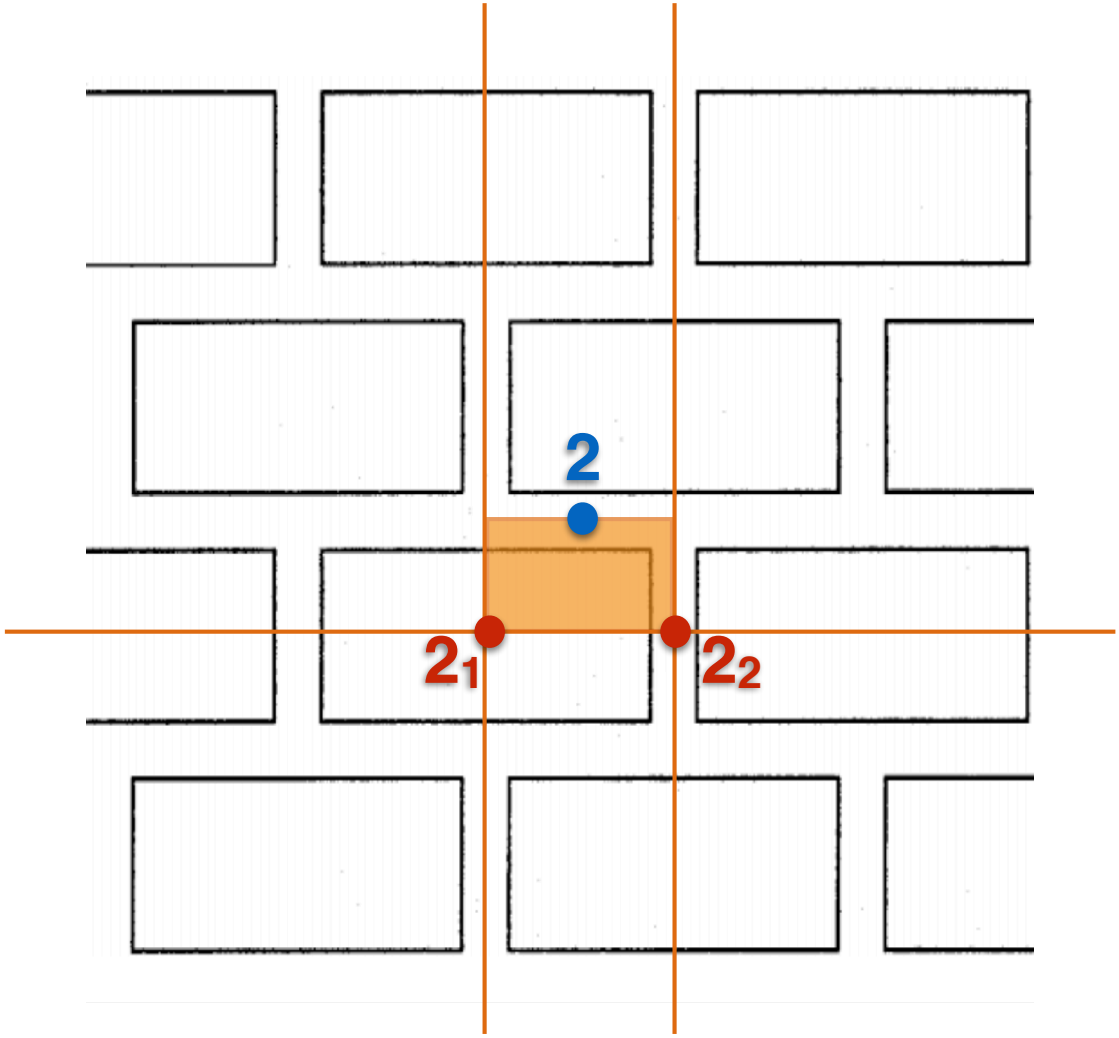
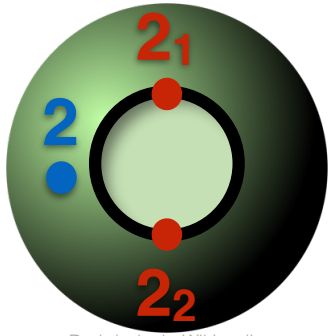
Symmetry groups and orbifolds

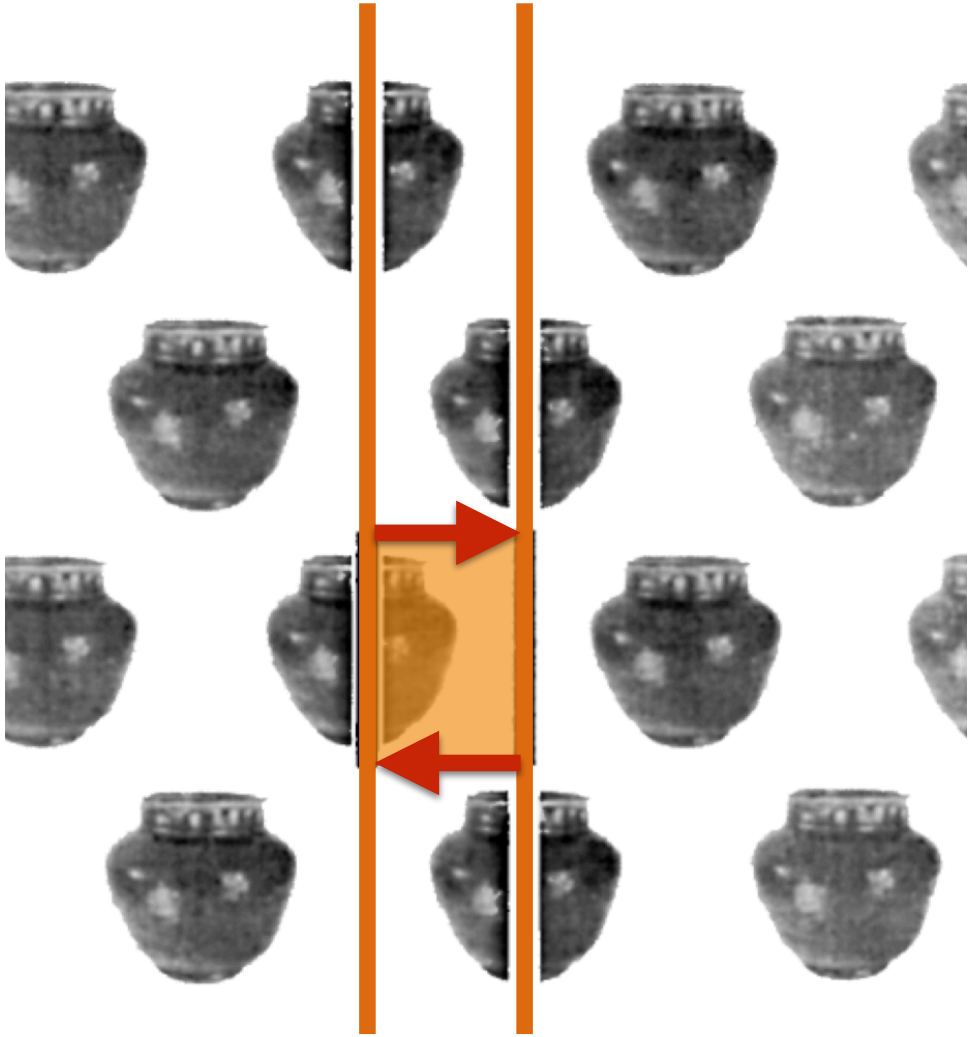


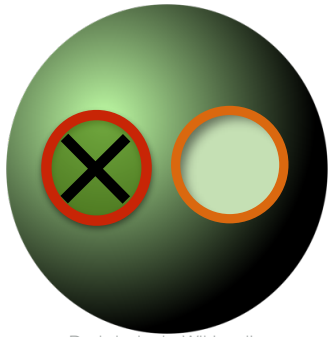


Darkdadaah, Wikipedia

2*22

Symmetry groups and orbifolds

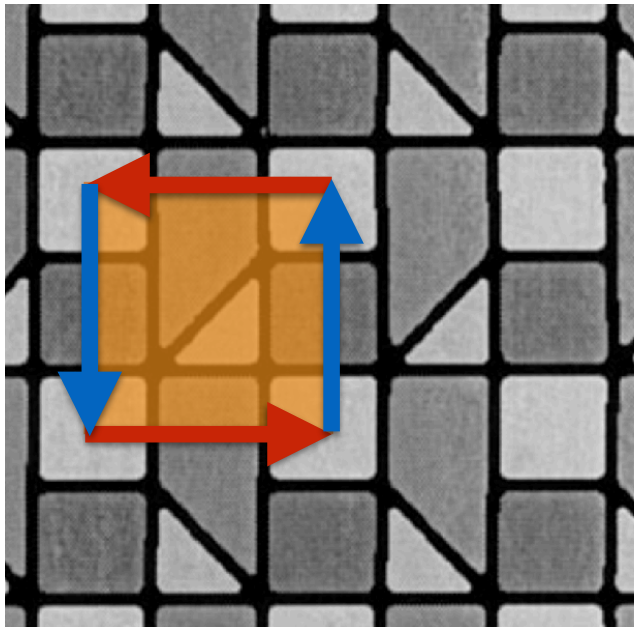


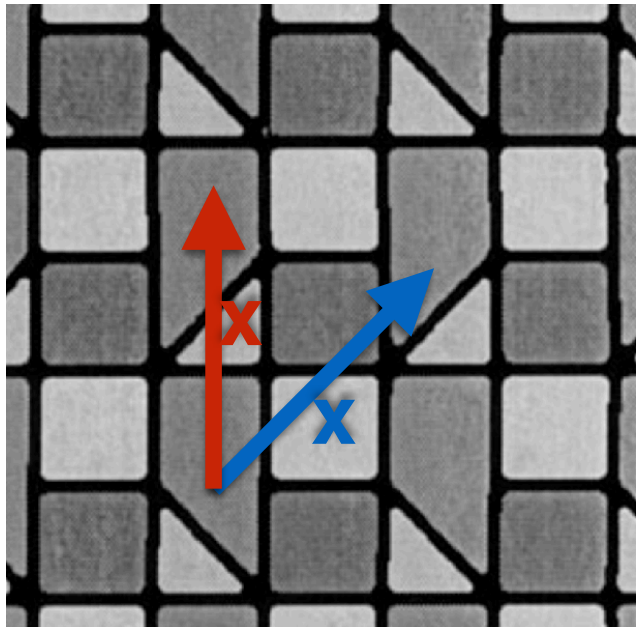


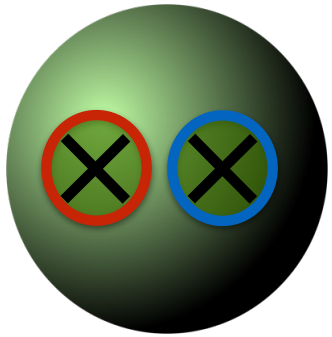
Darkdadaah, Wikipedia

*x

Symmetry groups and orbifolds







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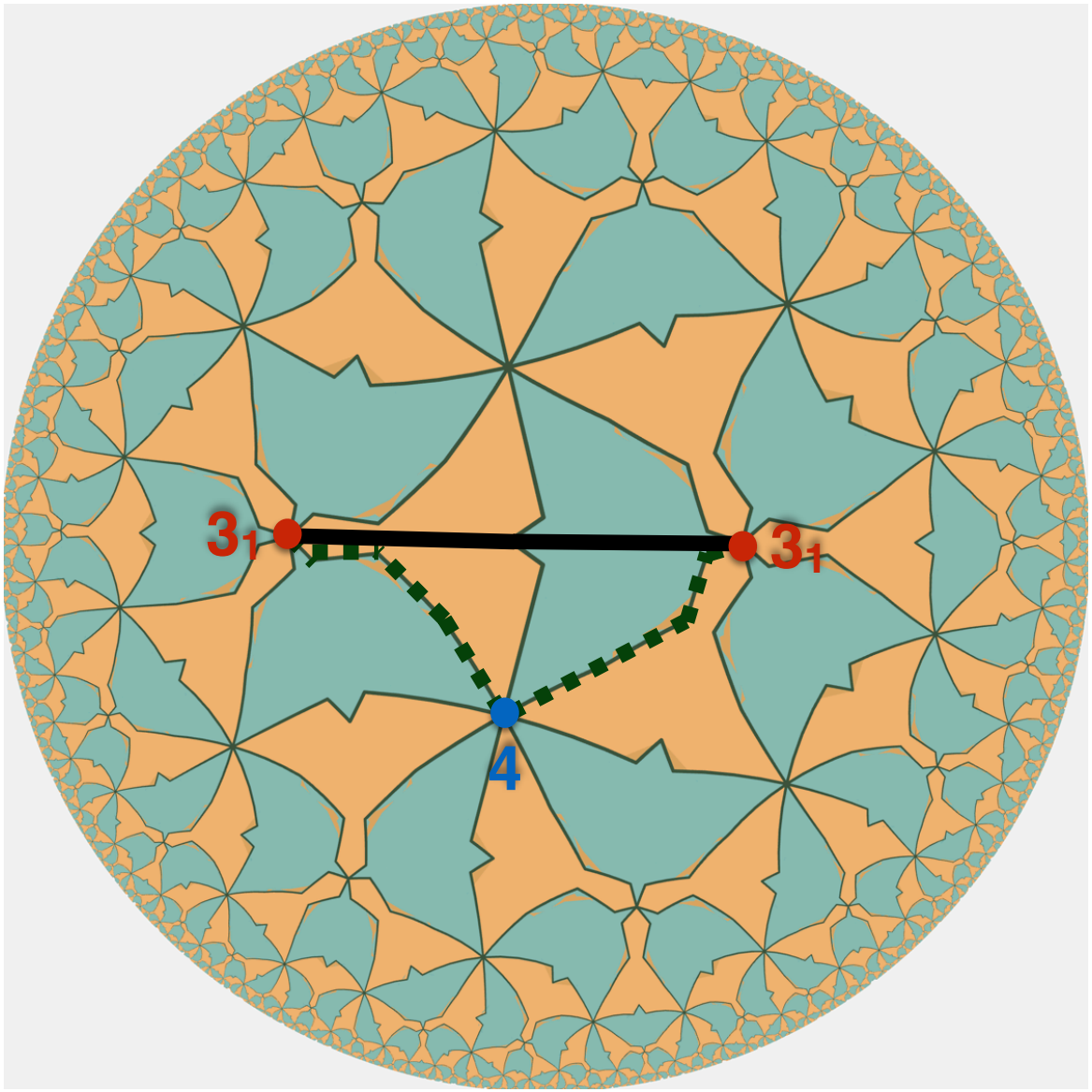
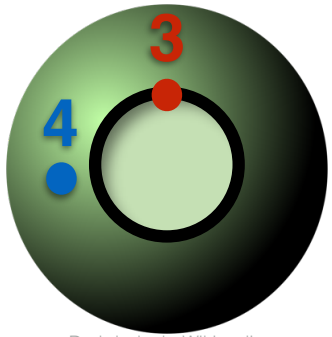
Symmetry groups and orbifolds





Darkdadaah, Wikipedia

4*3

Symmetry groups and orbifolds

Any 2D orbifold with symbol

o…o*ABC*…**abc*…**rpq*…*…x…x

can be obtained as

- $\mathbb{S}^2$ / an orthogonal group,
- $\mathbb{E}^2$ / a crystallographic group, or
- $\mathbb{H}^2$ / a NEC group,

except for the "bad orbifolds"

- *p*, *pq*, **p* and **pq* (with $p, q \geq 2$, $p \neq q$).

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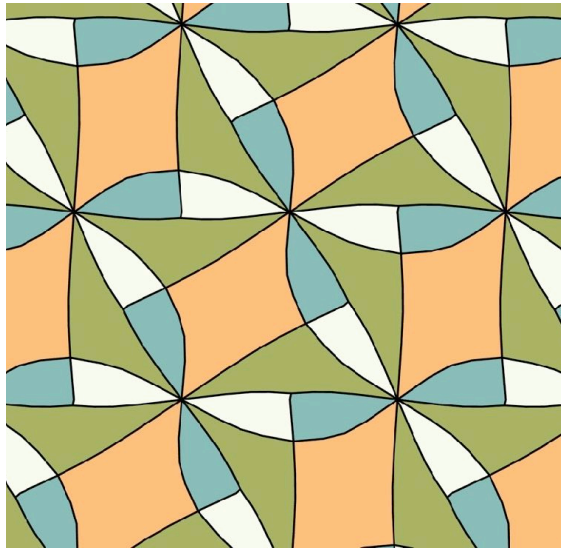
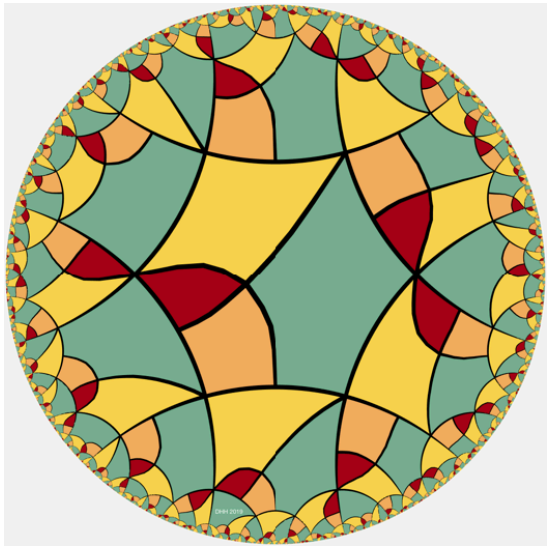
Periodic tilings

*532

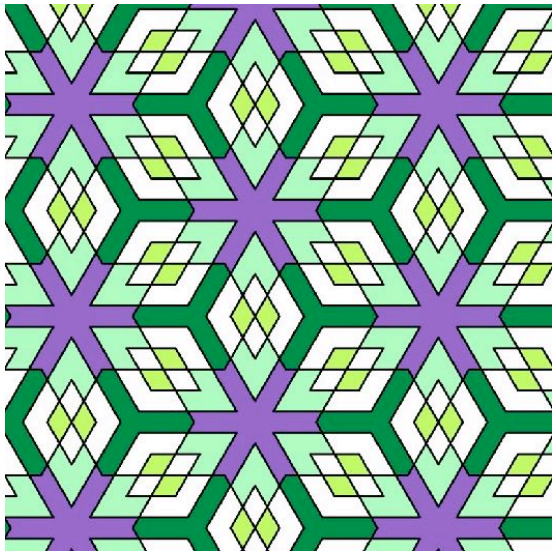


22x



2xx

*532



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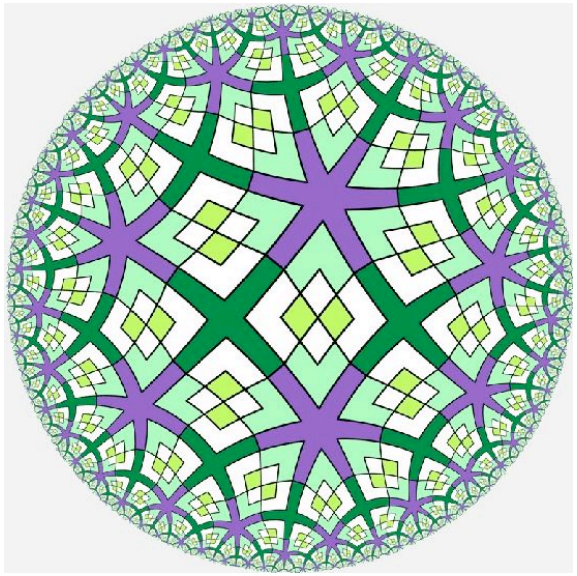

*642

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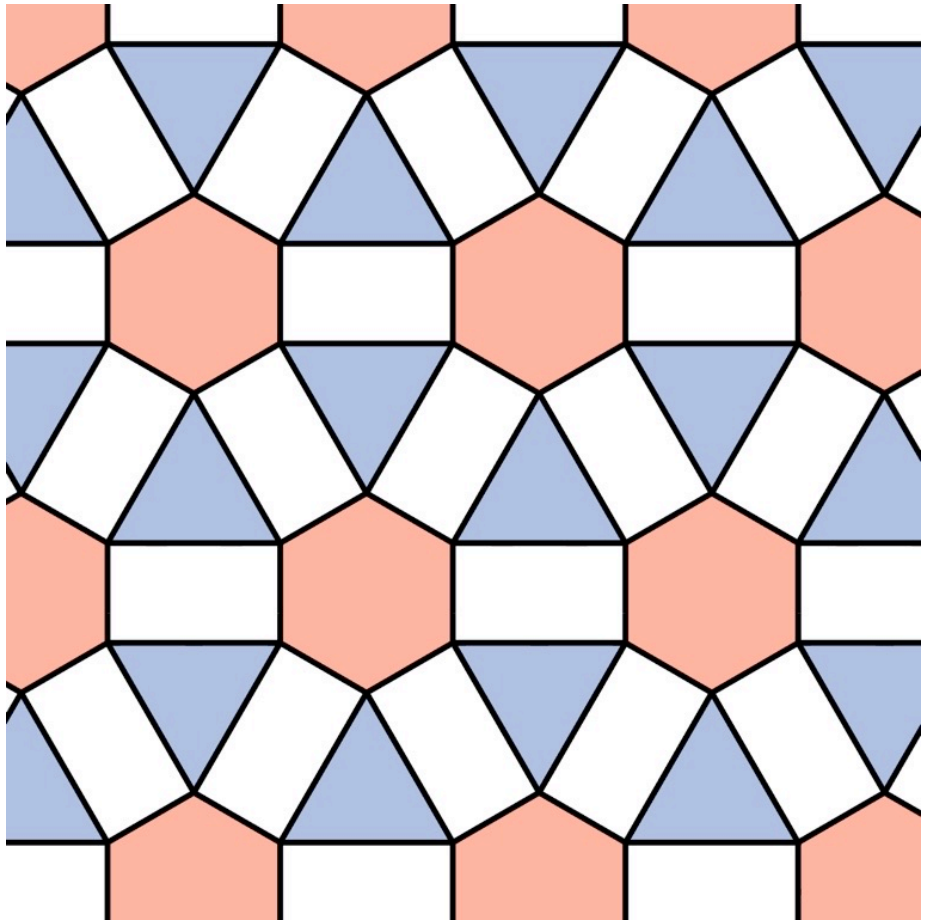
Equivariant tilings

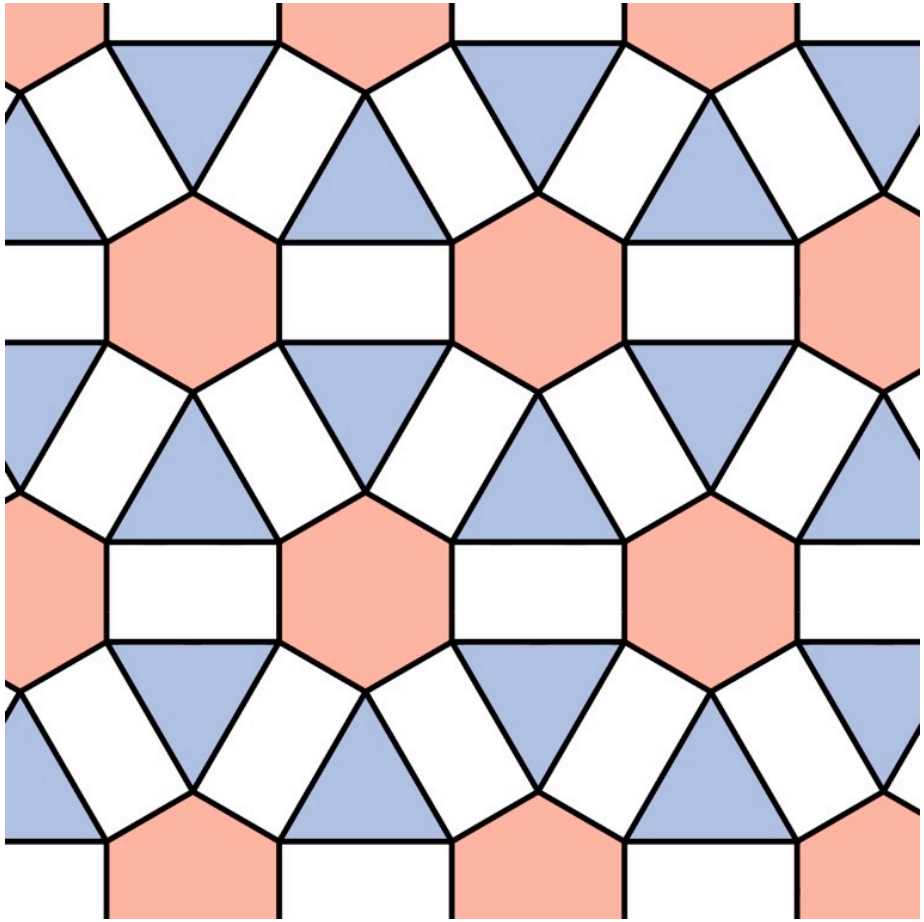
Equivariant tiling ($\mathcal{T}$,Γ):

- Tiling $\mathcal{T}$
- Prescribed symmetry group Γ



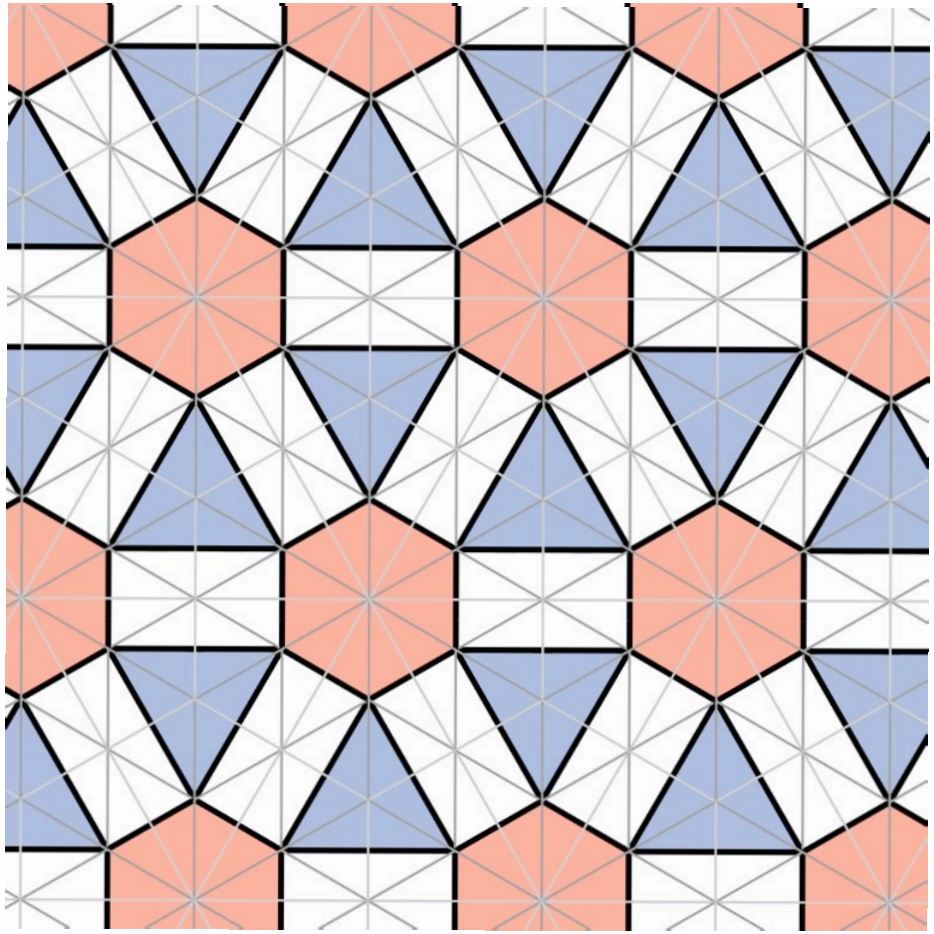
Combinatorics

- How to capture the combinatorial structure of such a tiling $(\mathcal{T},\Gamma)$?



Combinatorics

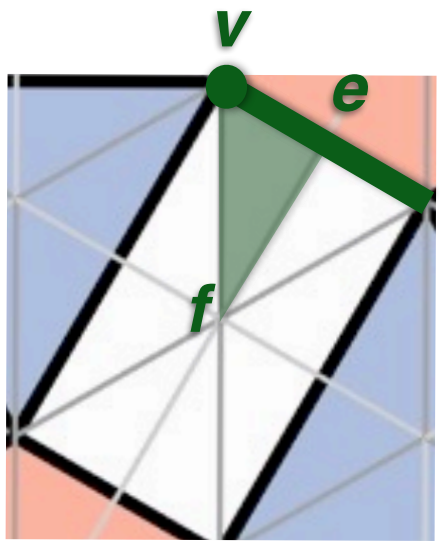
- Barycentric subdivision:



or:

set of chambers,

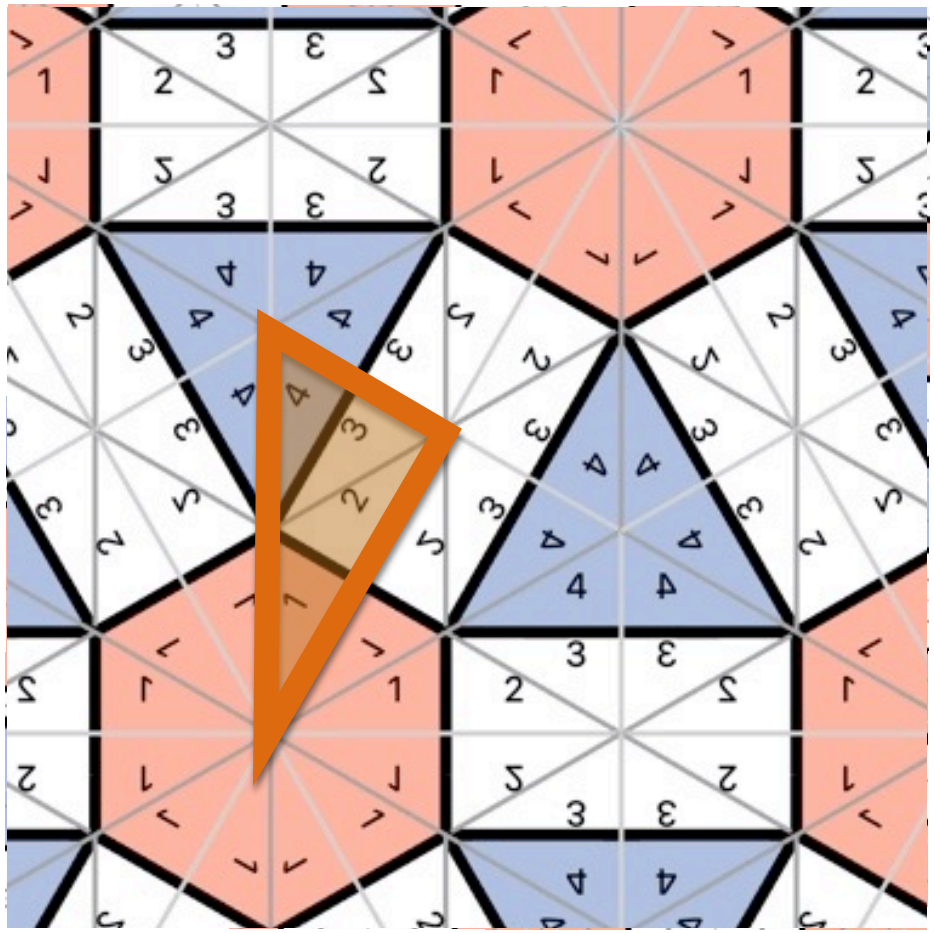
set of flags:

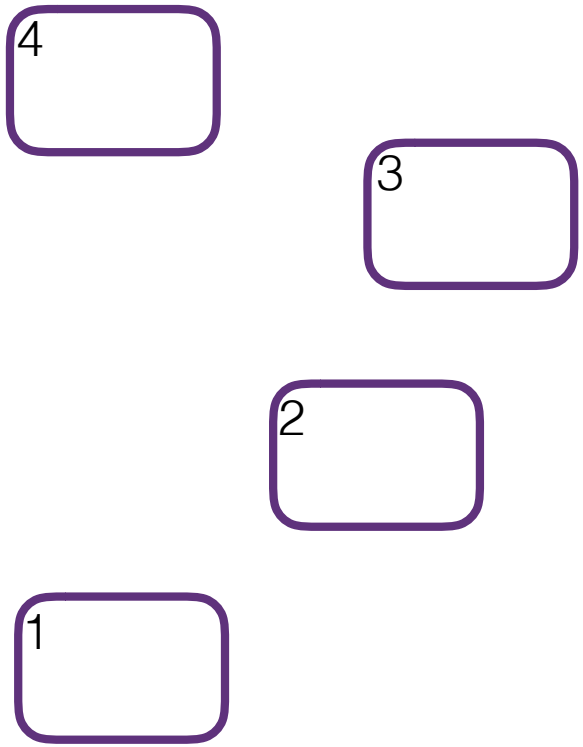


(v,e,f)

Combinatorics

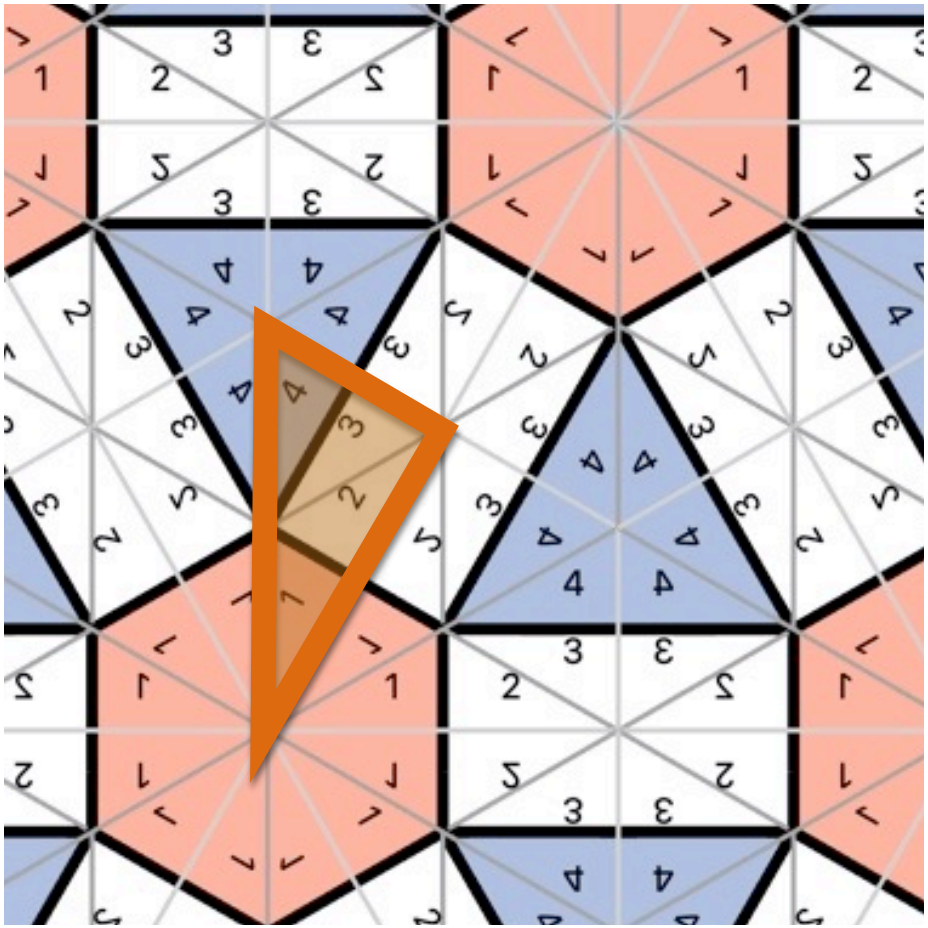
- Consider orbits under symmetry group Γ:

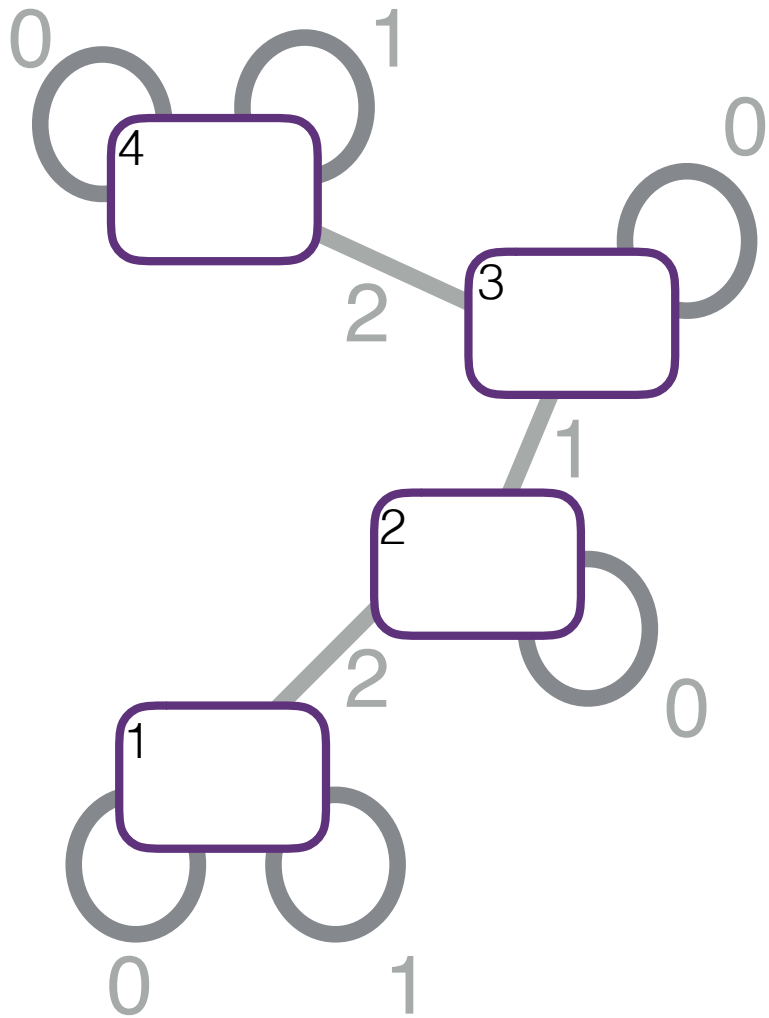




Combinatorics

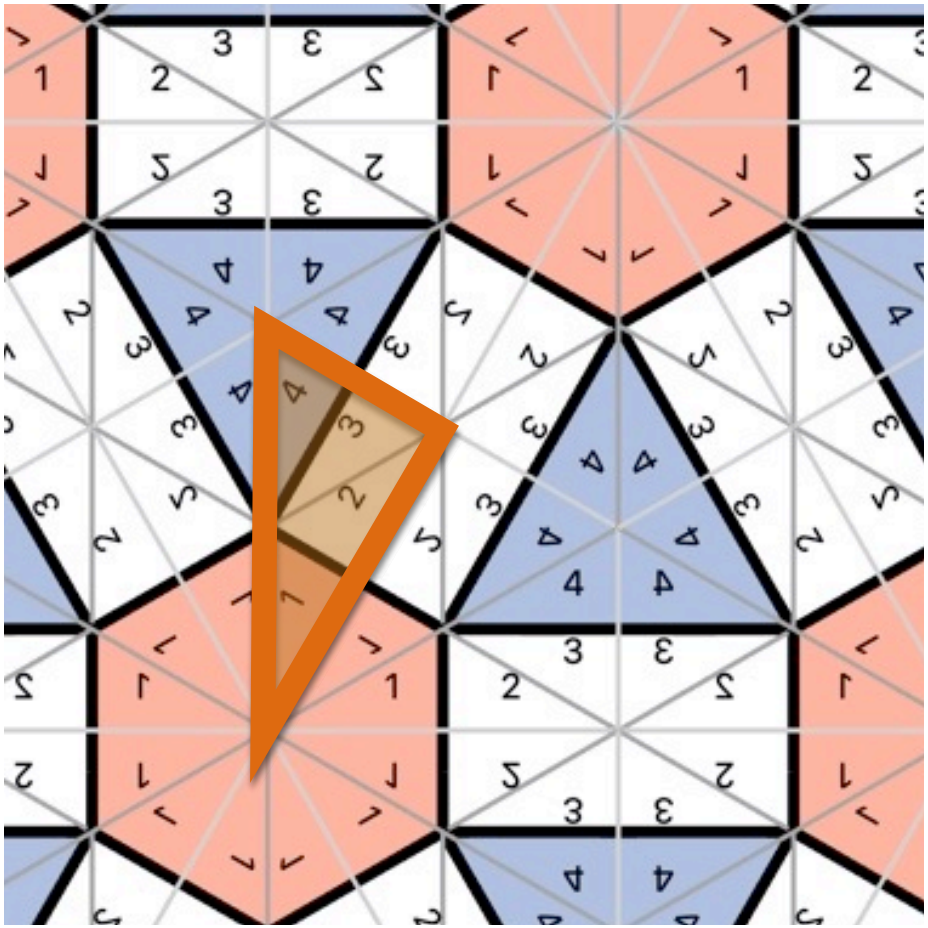
- Neighborhood relationships:

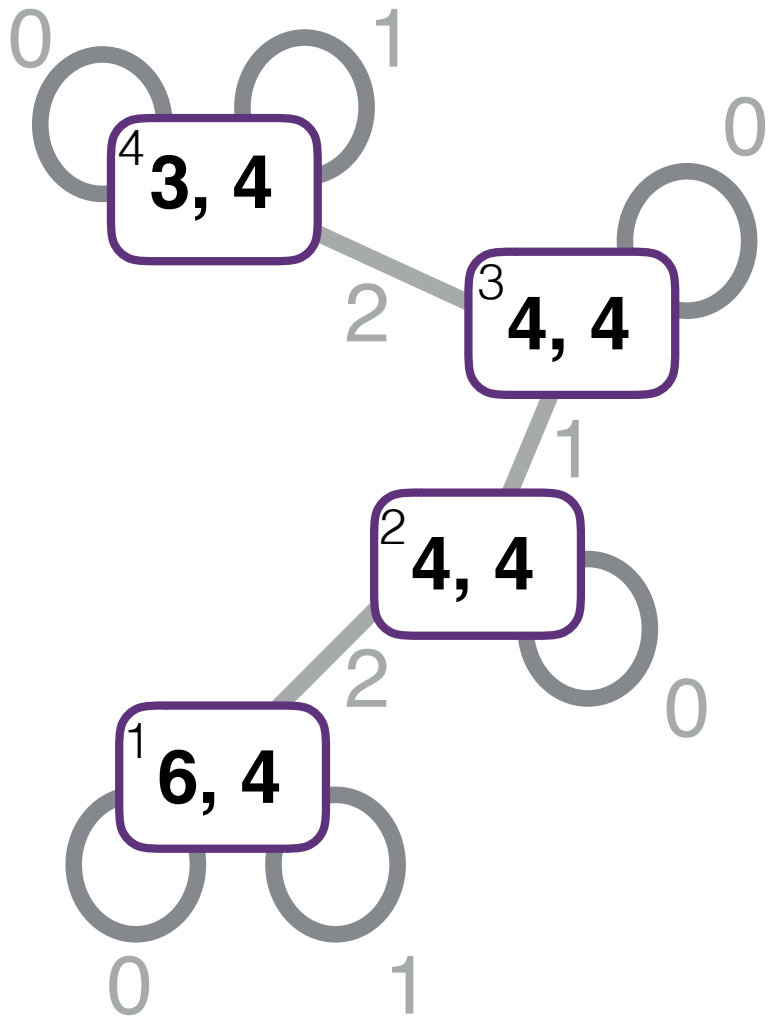




Combinatorics

- Face degrees and node degrees:





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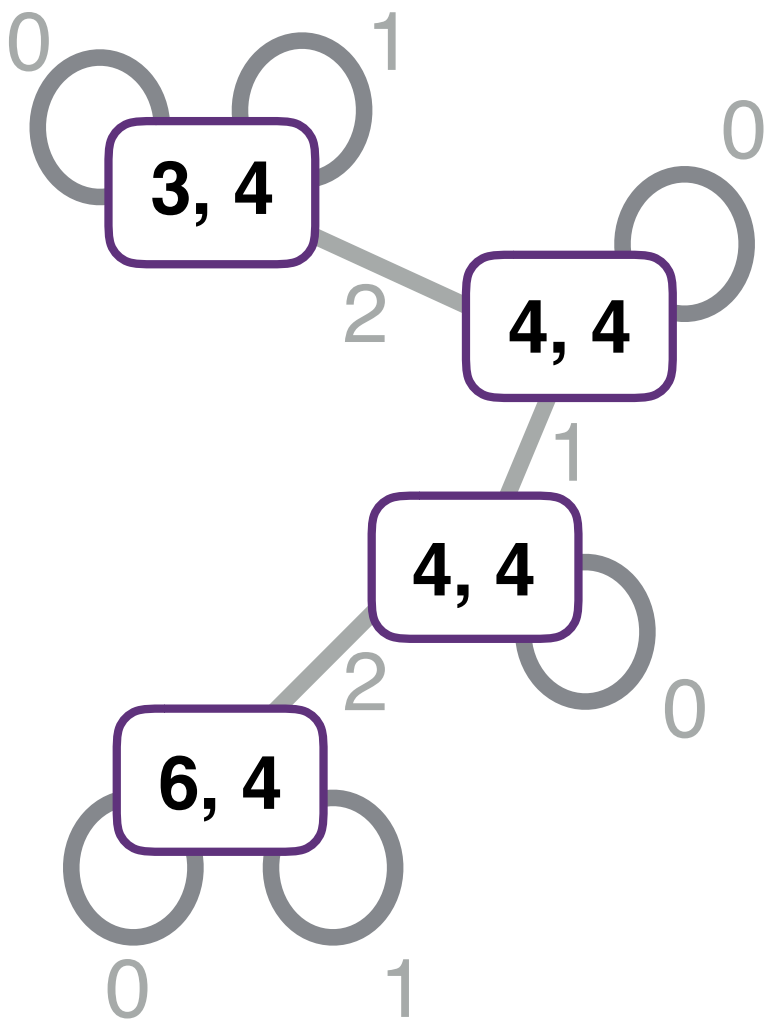
Delaney-Dress symbol



Delaney-Dress symbol ($\mathcal{D}$,m):

- $\mathcal{D}$ = set of nodes
- $\Sigma = \langle\sigma_0,\sigma_1,\sigma_2\rangle$ set of edges (involutions)
- face degrees
  - m01: $\mathcal{D} \rightarrow \{1,2,\ldots\}$
- node degrees:
  - m12: $\mathcal{D} \rightarrow \{3,4,\ldots\}$

+ conditions

A.W.M. Dress (1985) Regular polytopes and equivariant tessellations from a combinatorial point of view

Key Observation

Lemma (A.W.M. Dress)

Two equivariant tilings $(\mathcal{T}_1,\Gamma_1)$ and $(\mathcal{T}_2,\Gamma_2)$ are *equivalent*, iff their Delaney-Dress symbols $(\mathcal{D}_1,\mathrm{m}_1)$ and $(\mathcal{D}_2,\mathrm{m}_2)$ are *isomorphic*.

A.W.M. Dress and D.H.H. (1987) On tilings of the plane.

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UNIVERSITÄT
TÜBINGEN
Department of
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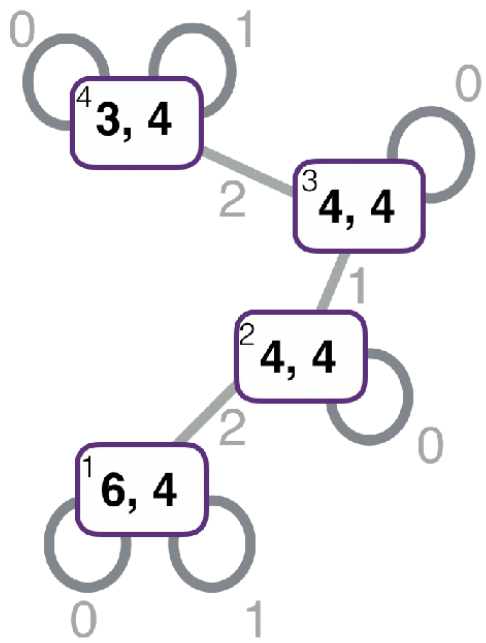
Simple properties

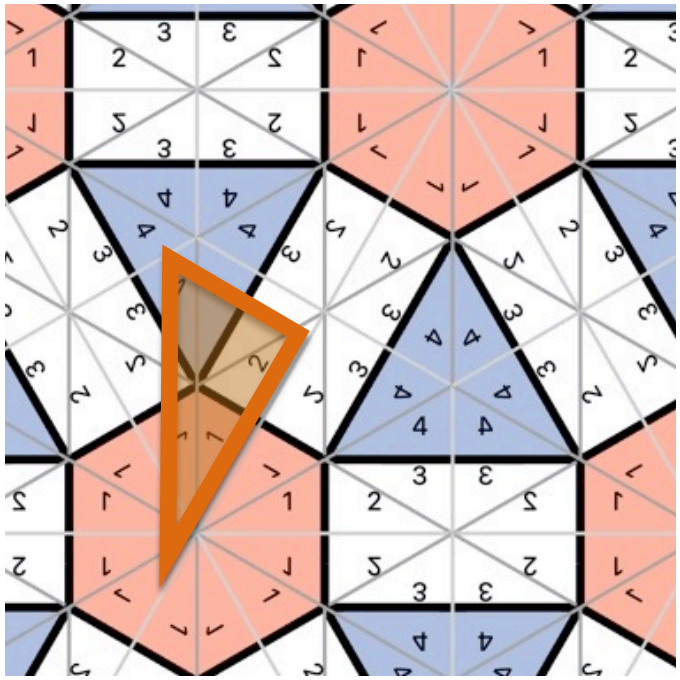
Equivalence classes of

- tiles: 0-1-components
- edges: 0-2-components
- vertices: 1-2-components



Here:

- tiles: 3
- edges: 2
- vertices: 1



Advanced properties

- Euler characteristic
- Curvature
- Geometry
- Orbifold name

Advanced properties

Curvature:

$$\mathscr{K}(\mathscr{D}, m) = \sum_{D \in \mathscr{D}} \left(\frac{1}{m_{01}(D)} + \frac{1}{m_{12}(D)} - \frac{1}{2} \right)$$

determines geometry:

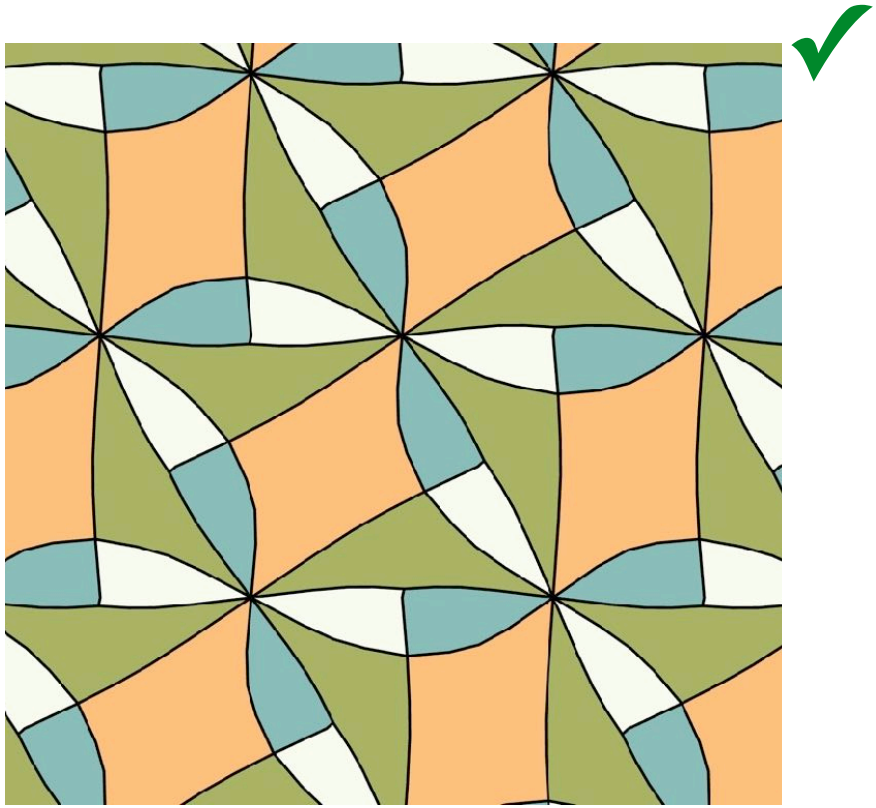
> 0: spherical

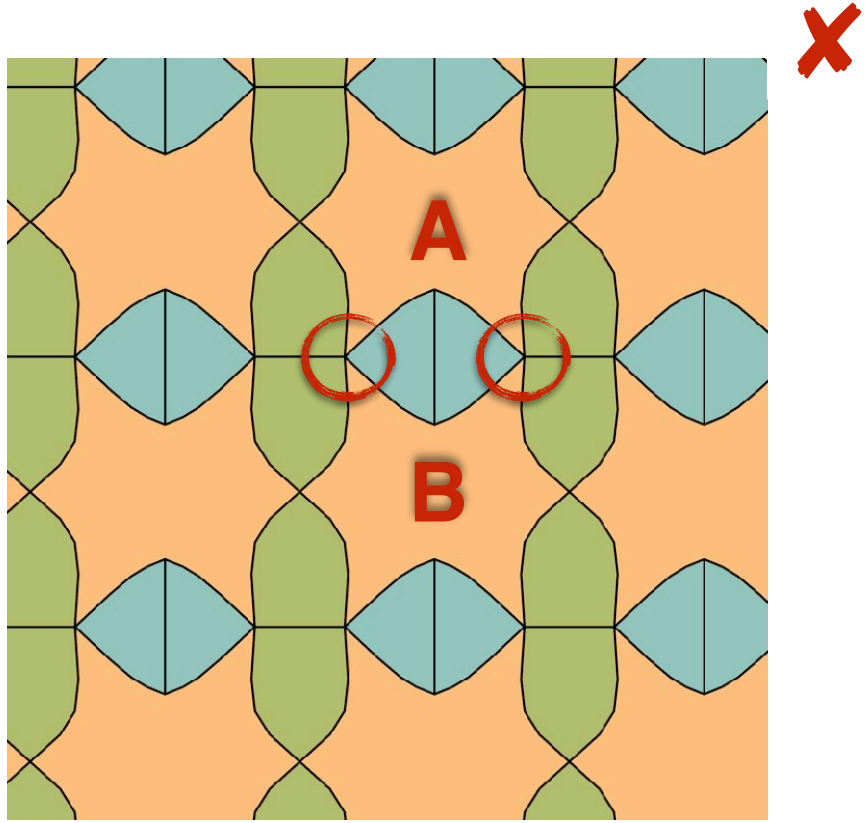
= 0: euclidean

< 0: hyperbolic

Difficult property:

- Tiling is *pseudo convex* if the intersection of any two tiles is either empty or connected:





Simplest property

- $|\mathcal{D}|$ “Dress complexity”

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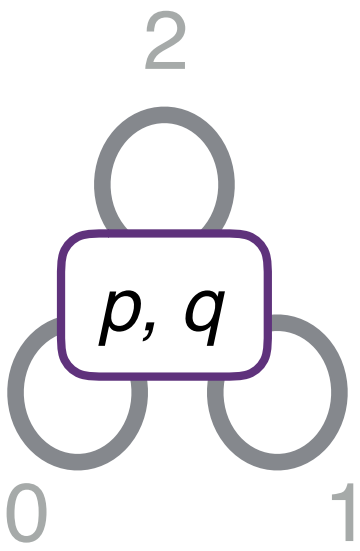
Dress complexity $|\mathcal{D}|=1$:

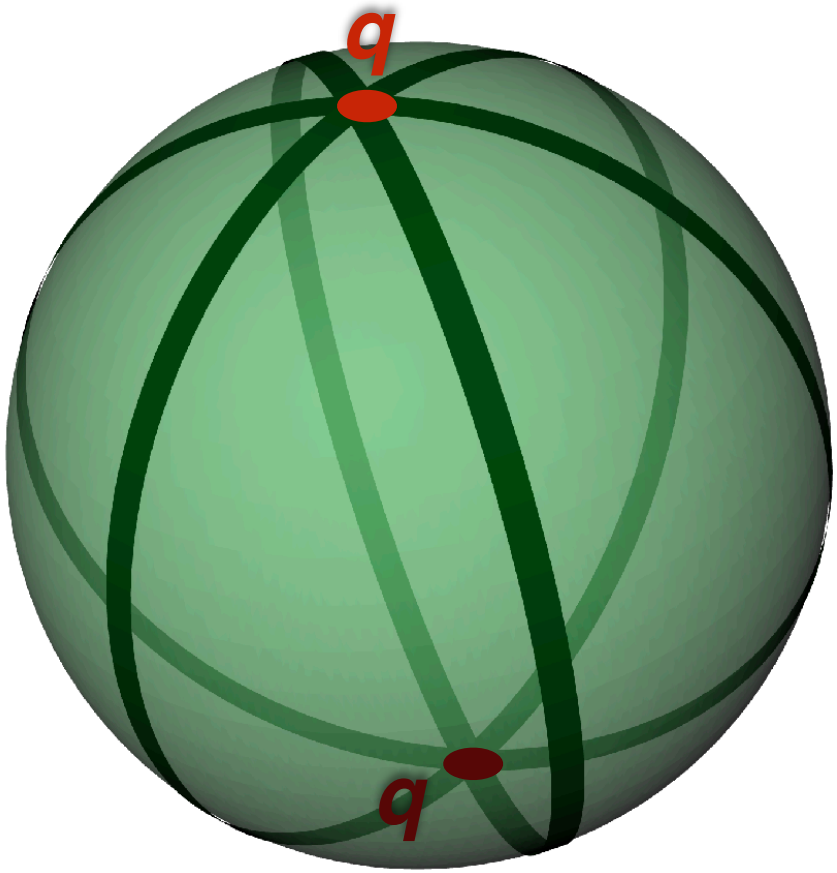


$p \geq 2,\ q \geq 3$

Dress complexity $|\mathcal{D}|=1$:

- p = 2: always spherical:

$$\mathcal{K}(\mathcal{D}, m) = \frac{1}{2} + \frac{1}{q} - \frac{1}{2} = \frac{1}{q} > 0$$



***q22**

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Dress complexity $|\mathcal{D}|=1$:

- $p = 3,\ q = 3$: $\mathcal{K}(\mathcal{D}, m) = \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6} > 0$
- We have: $q = 3, 4, 5$ spherical

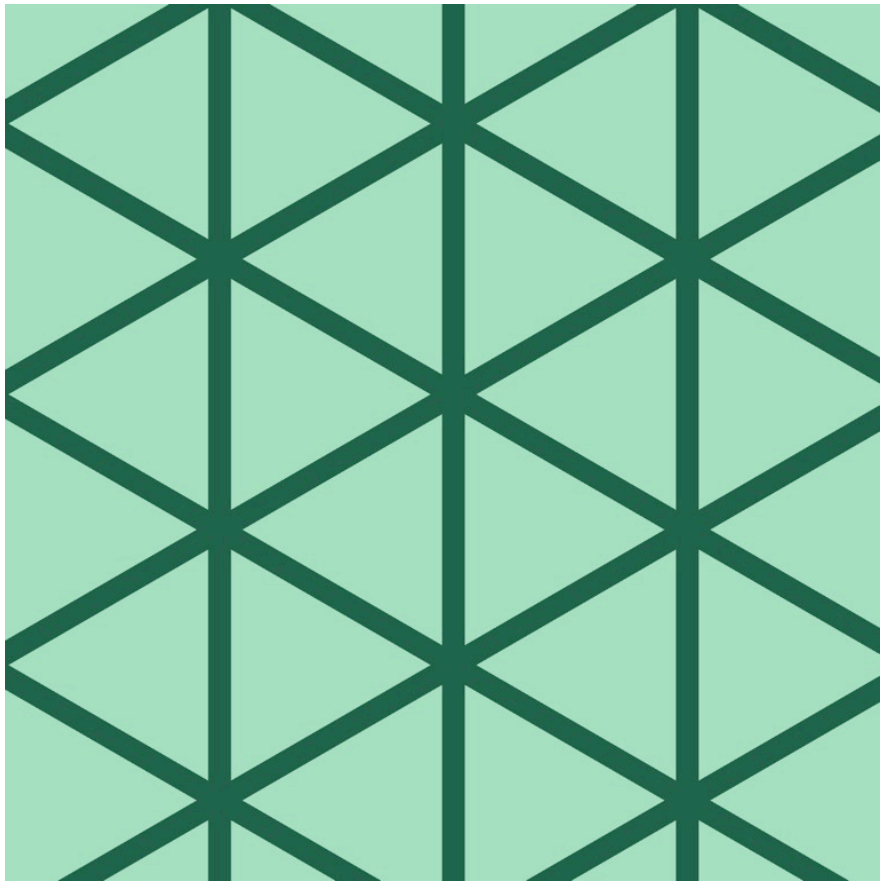
***332**

***432**

***532**

Dress complexity $|\mathcal{D}|=1$:

- $p = 3$, $q = 6$: $$\mathcal{K}(\mathcal{D}, m) = \frac{1}{3} + \frac{1}{6} - \frac{1}{2} = \frac{1}{6} = 0$$



***632**

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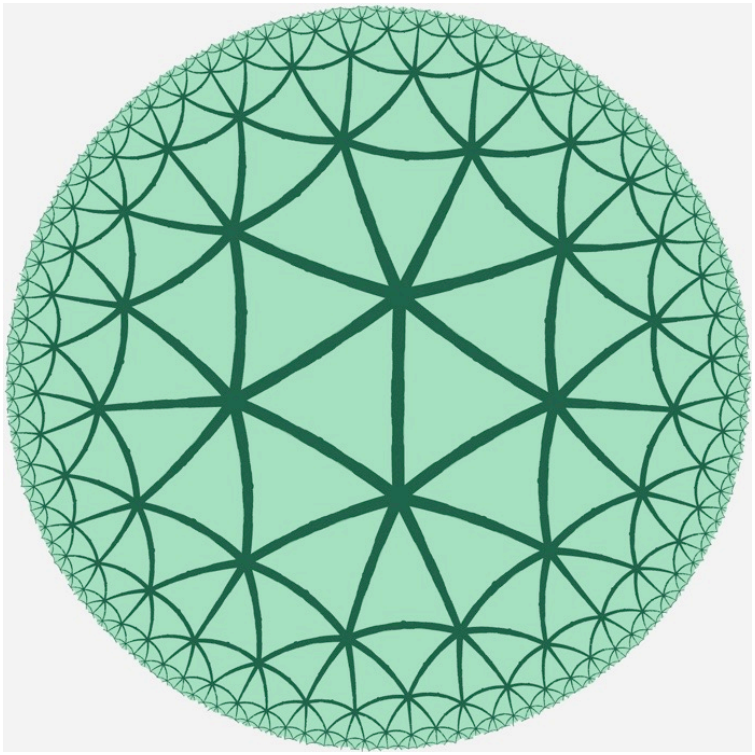
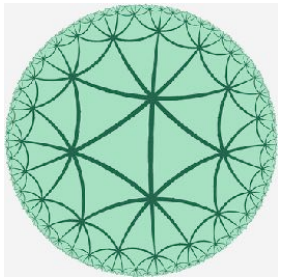
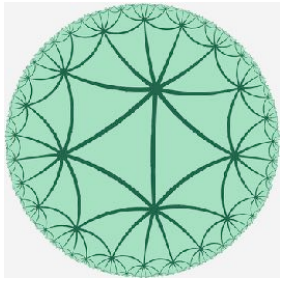
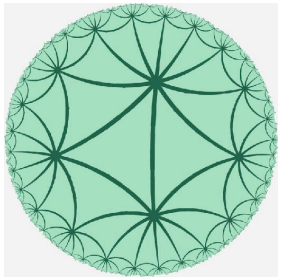
Dress complexity $|\mathcal{D}|=1$:

- $p = 3$, $q = 7$: $$\mathcal{K}(\mathcal{D}, m) = \frac{1}{3} + \frac{1}{7} - \frac{1}{2} < 0$$

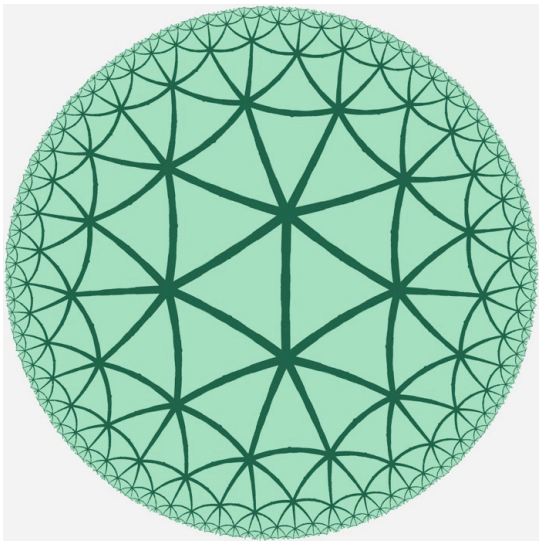


***732**

$q = 8$:  $q = 9$:  $q = 10$:  …

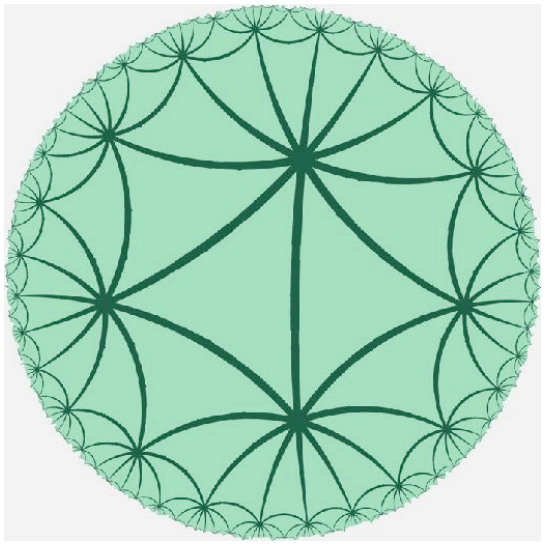
Geometry minimal

- $(\mathcal{D},m)$ is **geometry minimal** if either *spherical* with $v_{01}(D) \leq 5$ and $v_{12}(D) \leq 5$, or *euclidean*, or *hyperbolic* and can't reduce $v_{01}(D)$ or $v_{12}(D)$, for any $D \in \mathcal{D}$, without changing sign of curvature (i.e. geometry)



3, 7

***732**



3, 10

***(10)32**

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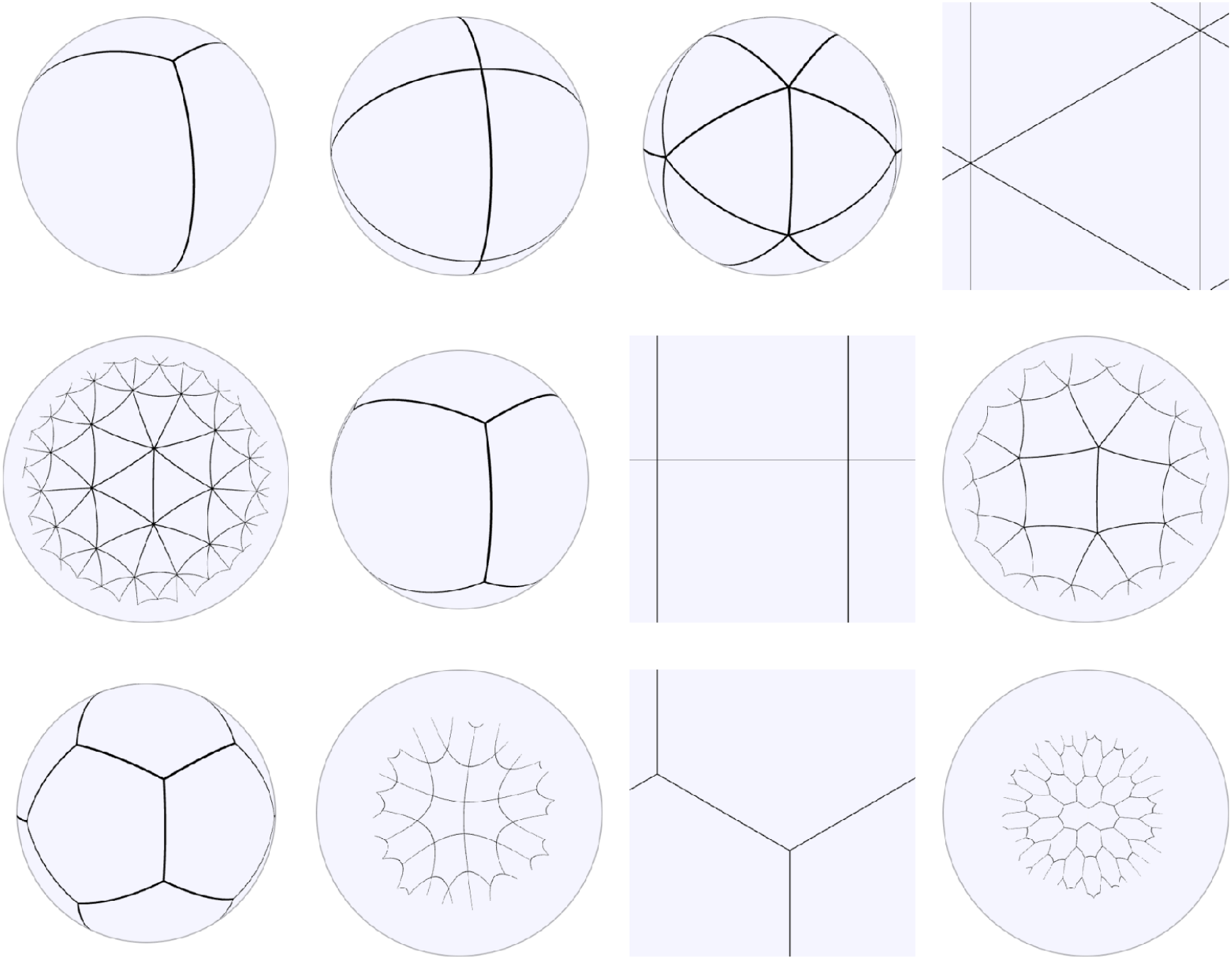
All geometry-minimal with $|\mathcal{D}|=1$: $(P \geq 3, Q \geq 3)$



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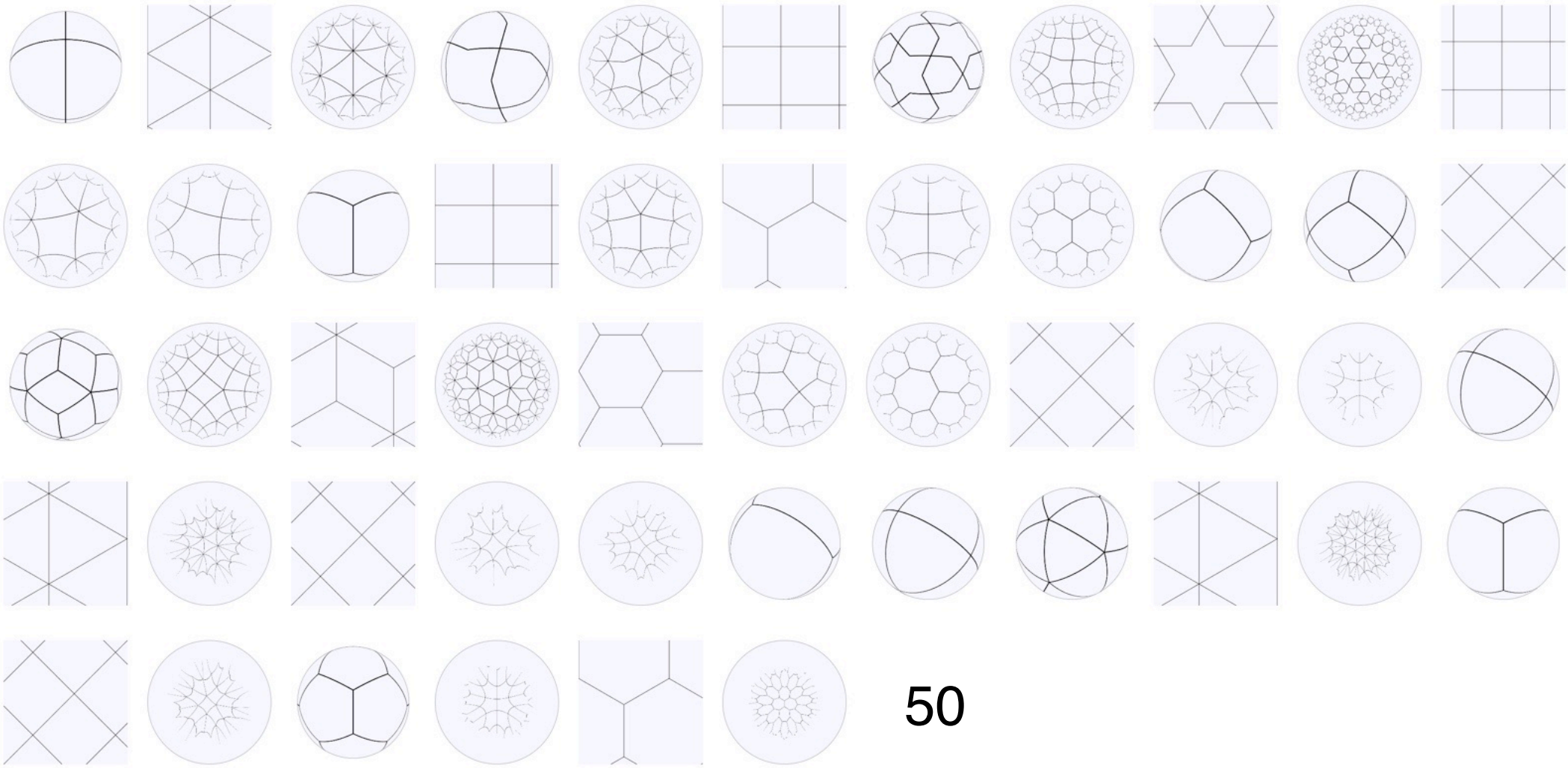
All geometry-minimal with $|\mathcal{D}|=2$: $(P \geq 3, Q \geq 3)$



50

A GALAXY OF PERIODIC TILINGS

- All geometry-minimal with $|\mathcal{D}| \leq 24$:

2,395,220,319

- of which:
  - 2,155,818 are spherical and
  - 1,728,488 euclidean.

Unpublished, with Olaf Delgado and Rüdiger Zeller

Contents

- Topology
- Geometry
- Combinatorics
- Algorithms and Software

Delaney-Dress symbol enumerator

- Orderly generation
- Program written in Julia
- Takes a few hours for $|\mathcal{D}| \leq 24$

Olaf Delgado

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UNIVERSITÄT
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Visualization and exploration

with Rüdiger Zeller

Tegula

- Interfaces database of Delaney-Dress symbols
- Supports complex queries
- Algorithm for constructing fundamental domain

  (Klaus Westphal, diploma thesis 1991)
- Algorithms for copying fundamental domain
- Euclidean, spherical and hyperbolic geometry
- User interaction

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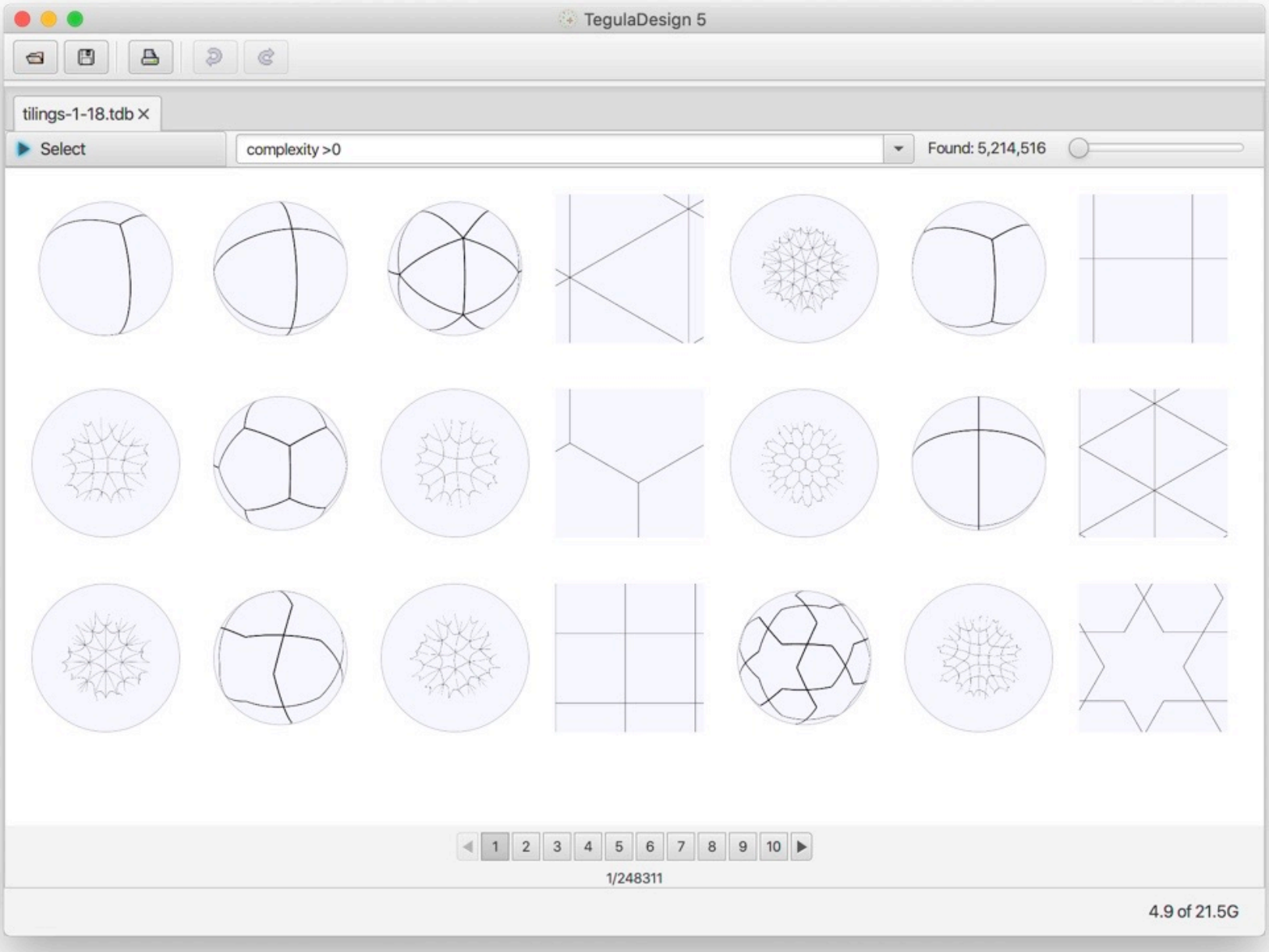

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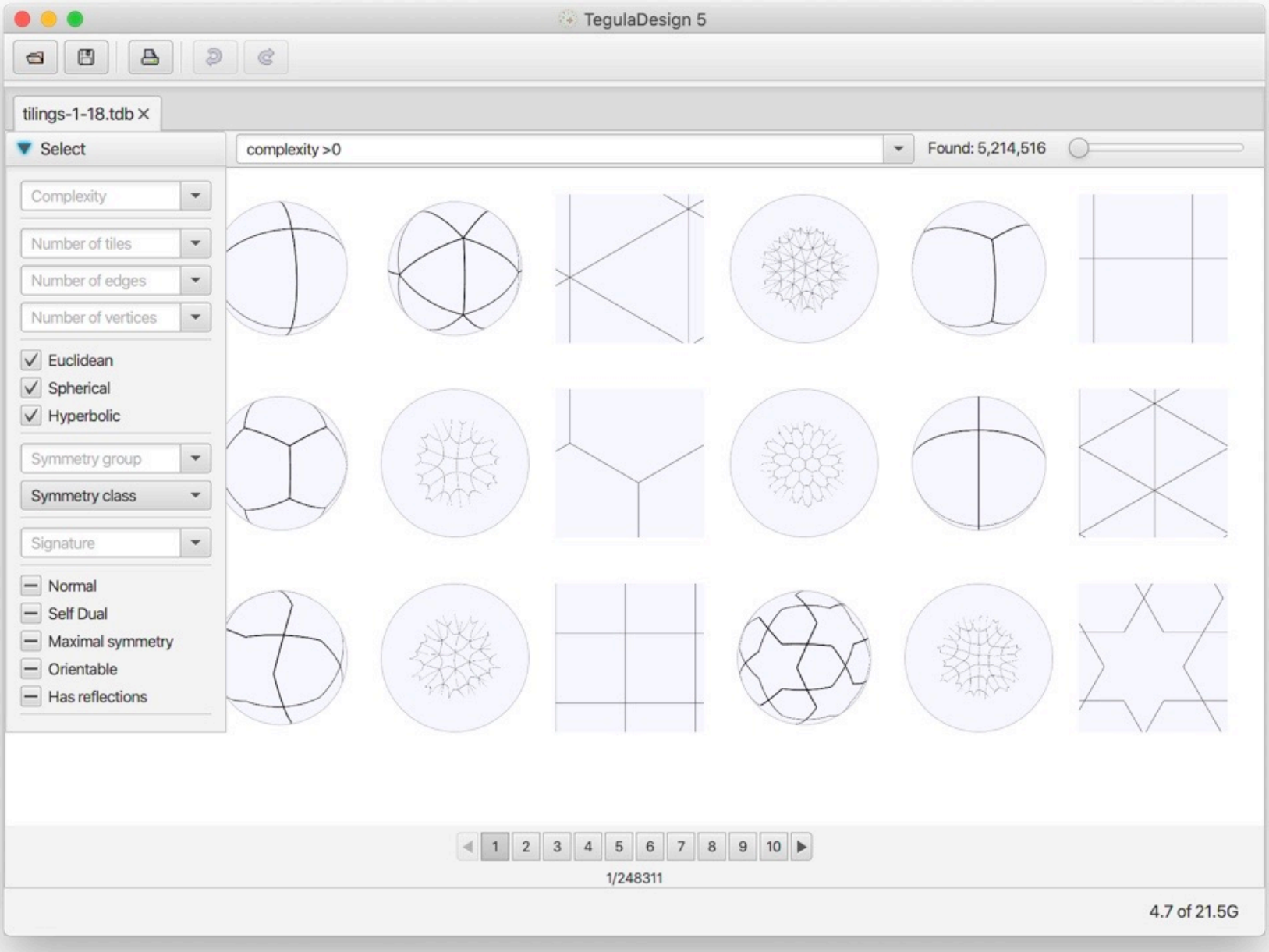
TegulaDesign 5
tilings-1-18.tdb ×
Select
complexity >0
Found: 5,214,516
Complexity
Number of tiles
Number of edges
Number of vertices
Euclidean
Spherical
Hyperbolic
Symmetry group
Symmetry class
Signature
Normal
Self Dual
Maximal symmetry
Orientable
Has reflections
1 2 3 4 5 6 7 8 9 10
1/248311
4.7 of 21.5G

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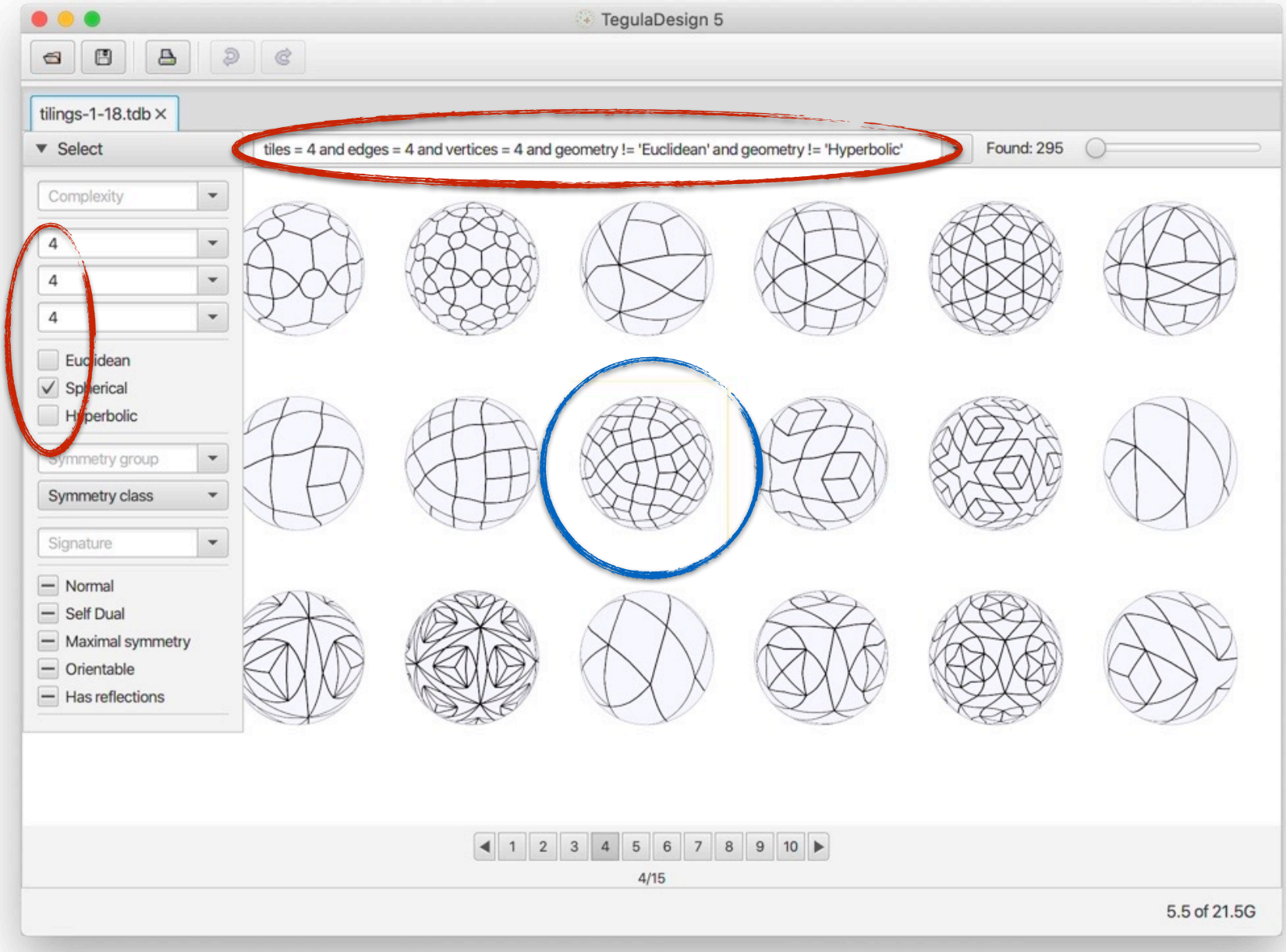

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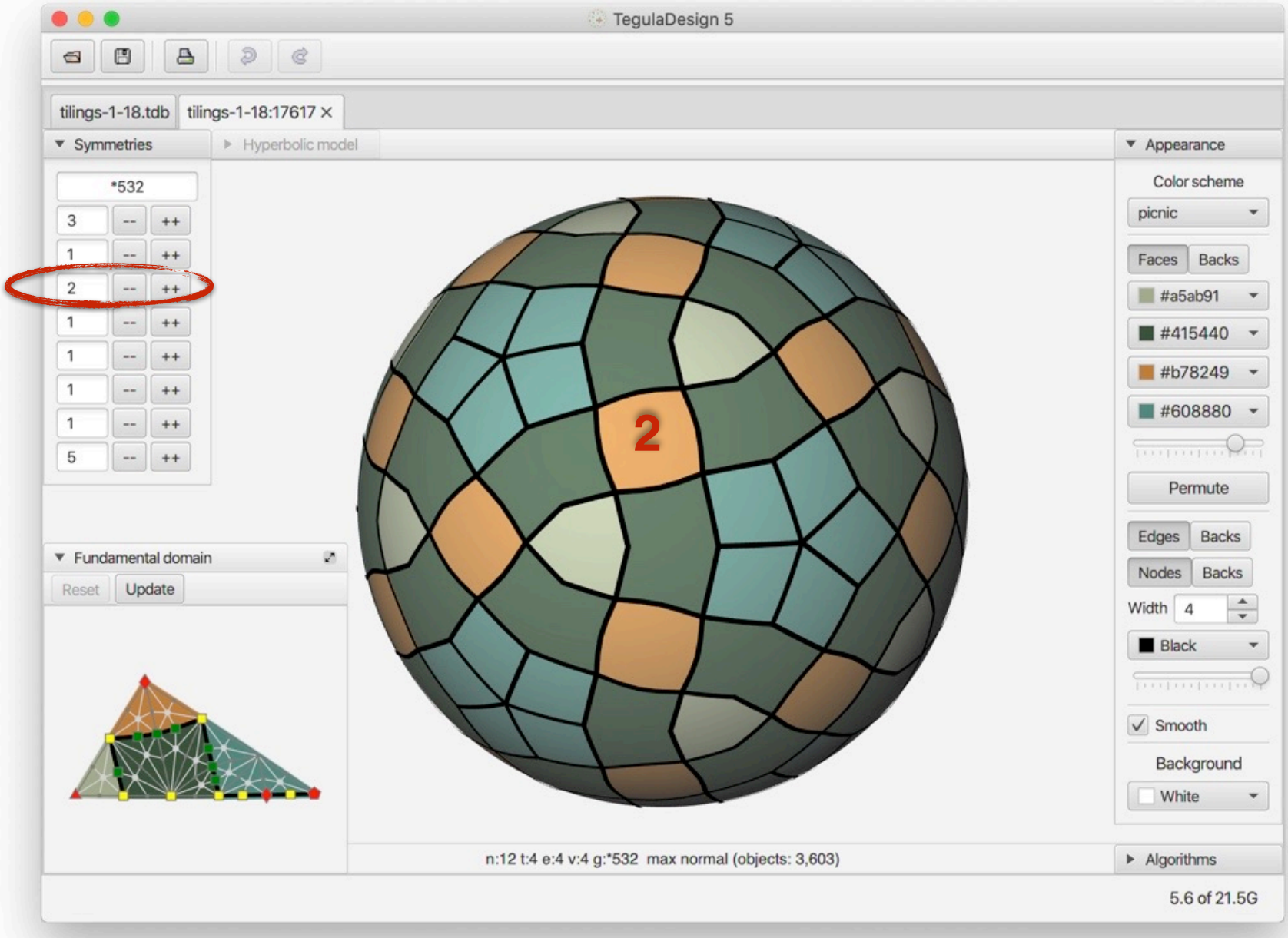

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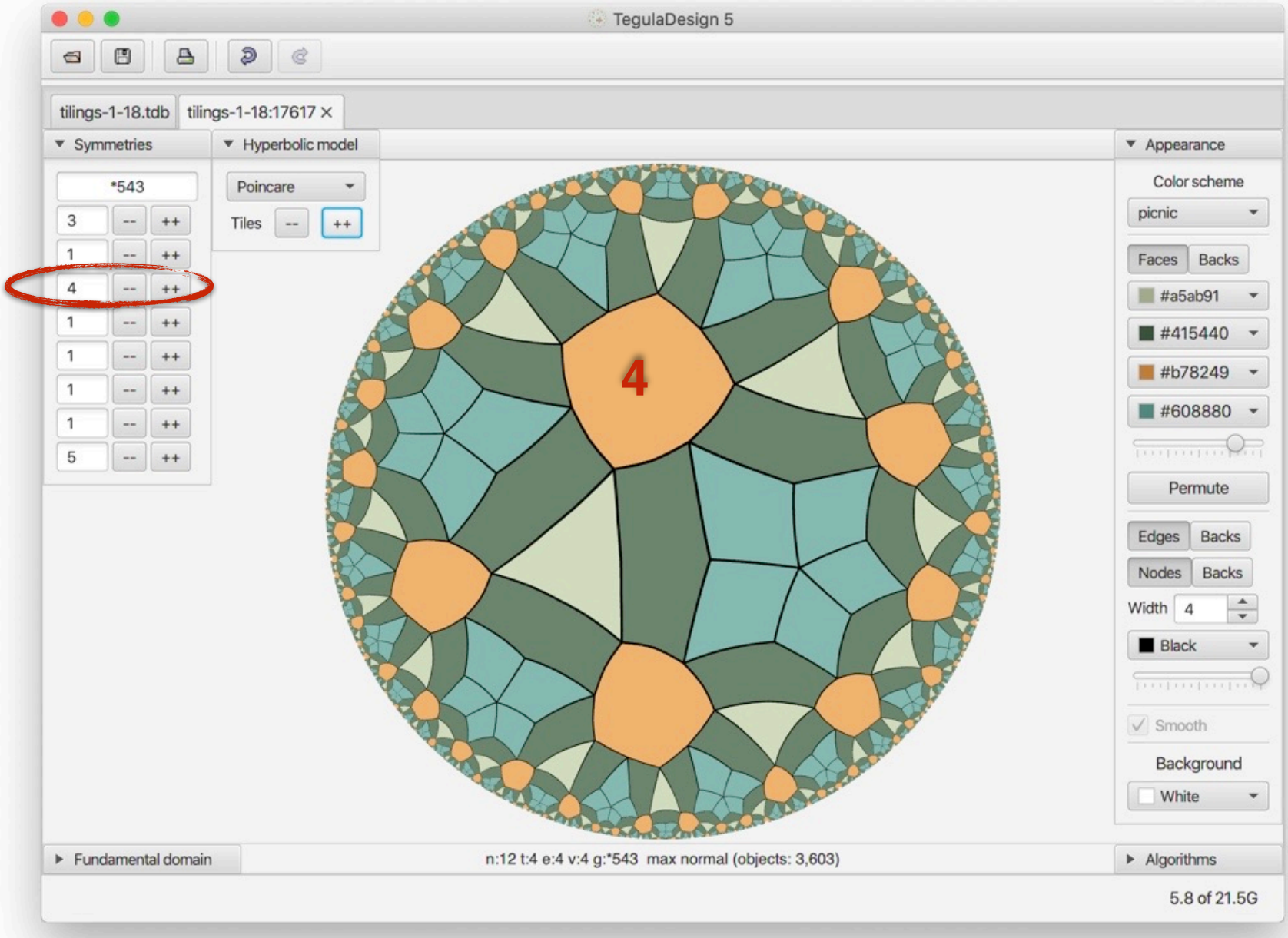

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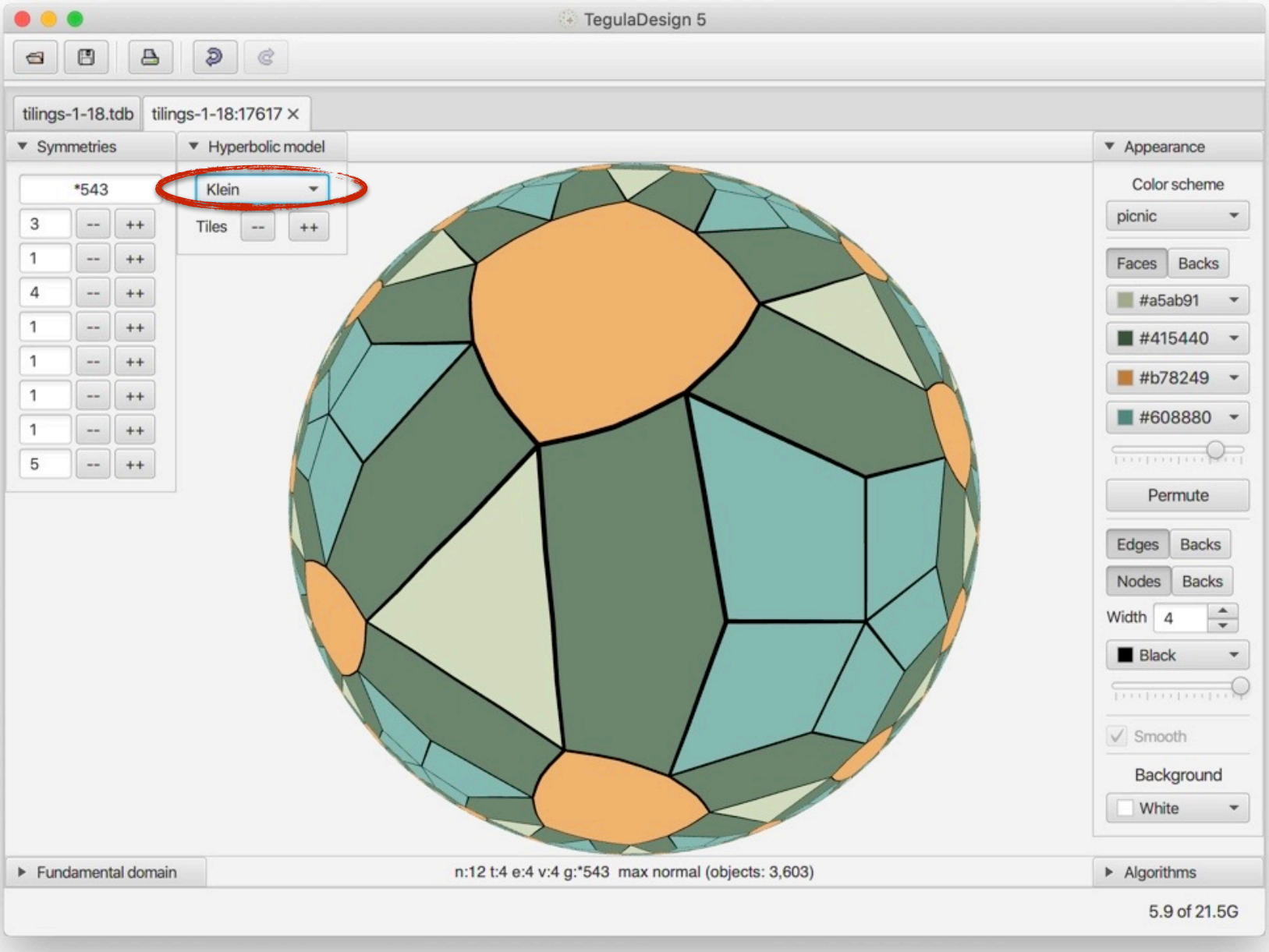
Tegula



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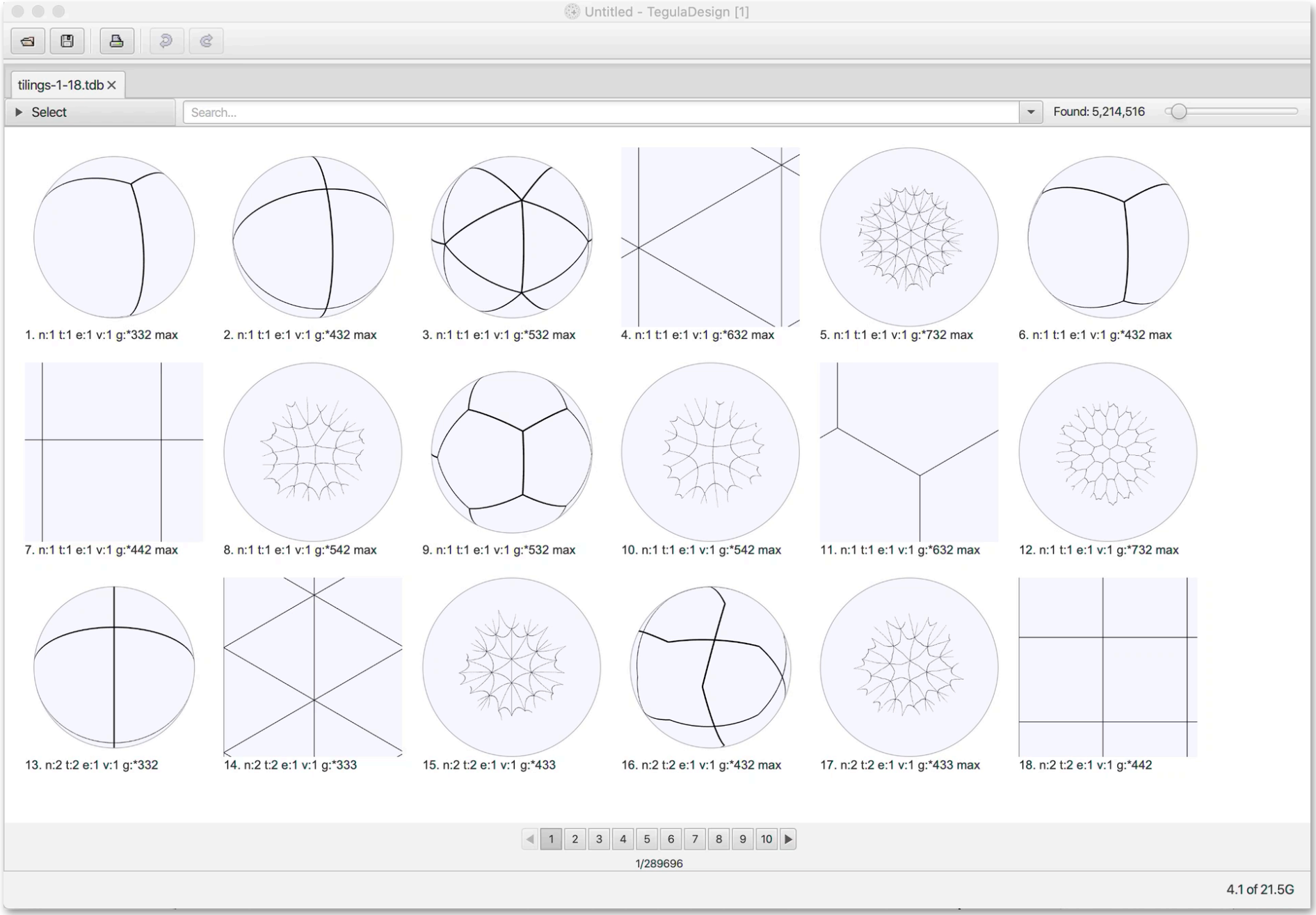
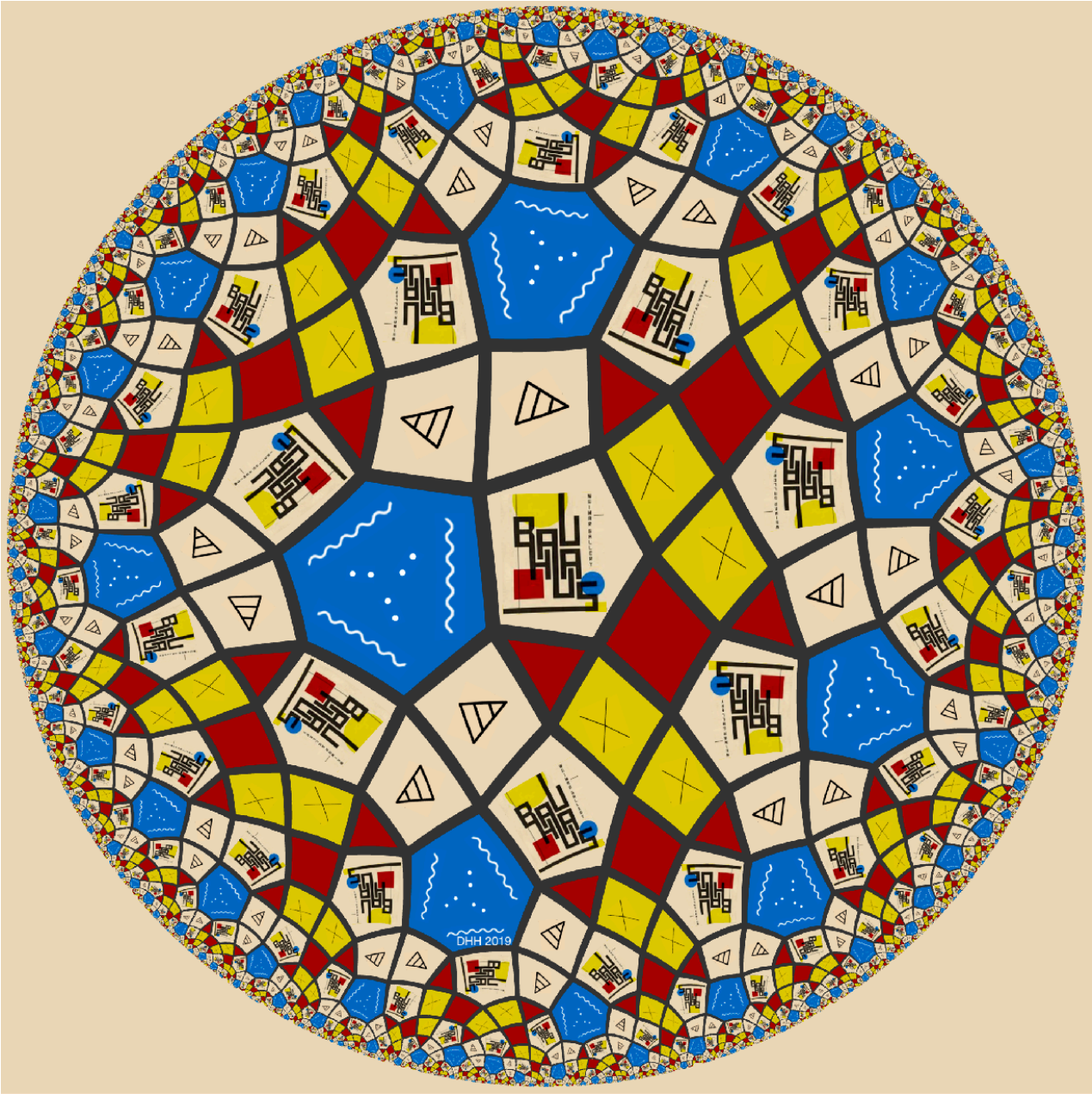
Using Tegula



TegulaDesign



Summary

- Orbifold notation for 2D symmetry groups
- Delaney-Dress symbols for equivariant tilings
- A galaxy of periodic tilings:
  - 2.4 billion 2D tilings with Dress complexity ≤ 24
- Tegula software

Acknowledgments

- Rüdiger Zeller
- Olaf Delgado
- Klaus Westphal
- Andreas Dress

Tegula runs on Linux, MacOS and Windows

tegula.husonlab.org